A Study of $W_3$ - Symmetric K-Contact Riemannian Manifold

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Abstract – In this paper the geometric properties of $W_3$ - curvature tensor are studied in K-contact Riemannian manifold.

Keywords – $W_3$ - Curvature Tensor, K-Contact Riemannian Manifold, Semi Symmetric, Symmetric and $W_3$ - Flat.

I. 2010 MATHEMATICS SUBJECT CLASSIFICATION: 53C15, 53C40

(1.0) Preliminaries:
Let $M_n$ be an $n (= 2m + 1) -$ dimensional contact Riemannian manifold with the structure tensors $(\Phi, T, A, g)$. Then the following formulas holds:

\[
\phi^2 X = -X + A(X)T, \quad A(T) = 1
\]

where $\phi = \frac{1}{2} L_T \phi$ where $h$ is the lie derivative, then any contact Riemannian manifold satisfies the condition that $h$ and $\phi$ are symmetric operators, $h$ ant-commutes with $\phi$ (i.e. $\phi h = h \phi = 0$), $A_{\phi} h = 0$ see[2] and [3] (1.4)

$F(X, Y) = g(X, \phi Y) - g(Y, \phi X) = (\nabla_X A)(Y) = -(\nabla_Y A)(X) = 0$ (1.3)

d$A(X, Y) = g(X, \phi Y)$ [see [1]] for any vector fields $X$ and $Y$ in M. If we defined an operator $h$ by $h = \frac{1}{2} L_T \phi$ where $h$ is the lie derivative, then any contact Riemannian manifold satisfies the condition that $h$ and $\phi$ are symmetric operators, $h$ ant-commutes with $\phi$ (i.e. $\phi h = h \phi = 0$), $A_{\phi} h = 0$ see[2] and [3] (1.4)

$\nabla_X T = -\phi X$ also in K-contact we have

(1.5)

(1.6)

(1.7)

(1.8)

(1.9)

where $R$ is the Riemannian (0,4) curvature tensors

\[S(T, T) = Ric(T, T) = n - 1,\]

(1.10)

then using the definition of $W_3$ - tensor in K-contact Riemannian Mishra and Pokhariyal [4] gave the definition of $W_3$ -tensor as

\[W_3(X, Y) Z = R(X, Y) Z + \frac{1}{n-1} [g(Y, Z) Q X - Ric(X, Z) Y]
\]

where $Q$ is the symmetric endomorphism of tangent space at a point to the Ricci tensor or

$W_3(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{n-1} [g(Y, Z) Ric(X, U) - g(Y, U) Ric(X, Z)]$

II. $W_3$-Curvature Tensor in K-Contact Riemannian Manifold

A K-contact Riemannian manifold is said to be $W_3$-flat if $W_3(X, Y) Z = 0$.

(2.1) Theorem:
A $W_3$ - flat K-contact Riemannian Manifold is a Space of negative scalar curvature, that is: $r = -n(n + 1)$

Put $W_3 = 0$

\[
\Rightarrow 0 = R(X, Y, Z, U) + \frac{1}{n-1} [g(Y, Z) Ric(X, U) - g(Y, U) Ric(X, Z)]
\]

Then using the definition of $Ric(X, Y) = (n - 1)g(X, Y)$ in the above equation we get

\[R(X, Y, Z, U) = \frac{1}{n-1} [n(n - 1) g(Y, U) g(X, Z) - (n - 1) g(Y, Z) g(X, U)]
\]

\[= [g(Y, U) g(X, Z) - g(Y, Z) g(X, U)] - [g(Y, U) g(X, Z) + g(Y, Z) g(X, U)]
\]

\[= -[g(Y, U) g(X, Z) + g(Y, Z) g(X, U)]
\]

\[= -[g(Y, U) g(X, Z) - g(Y, Z) g(X, U)]
\]


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III. $W_3$ - Symmetric K-Contact Riemannian Manifold

A K-contact Riemannian manifold is said to be symmetric if
\[(3.1) \quad \nabla_\mathcal{Y} W_3(X, Y, Z) = W_3'(X, Y, Z, U) = 0.\]

**Theorem 3:**
A $W_3$-symmetric and $W_3$-flat K-contact manifold is a flat manifold i.e. zero curvature.

**Proof:**
From the symmetric property it follow
\[(3.1.1) \quad R(X, Y, W_3(Z, U, V)) - W_3(Z, R(X, Y, U, V)) = 0\]

We expand the above equation
\[(3.1.2) \quad R(X, Y, W_3(Z, U, V)) = g(Y, W_3(Z, U, V))X - g(X, W_3(Z, U, V))Y\]
\[= W_3'(Y, Z, U, V)X - W_3'(X, Z, U, V)Y\]
\[= R(X, Y, W_3(Z, U, V)) = W_3'(I, Z, U, V) = 0\]

Hence the theorem

IV. $W_3$ - Semi Symmetric K-Contact Riemannian Manifold

A K-contact Riemannian manifold is said to be a $W_3$-semi symmetric if

\[(3.1.5) \quad W_3'(Z, U, R(X, Y, V), T) = R'(Z, U, R(X, Y, V), T)\]
\[+ \frac{1}{n-1} [g(U, R(X, Y, V))Ric(T, Z) - g(U, T)Ric(R(X, Y, V), Z)]\]
\[= R(Z, U, R(X, Y, V), T) + \frac{n-1}{n-1} [g(Z)R'(X, Y, V) - g(U)R'(X, Y, Z)]\]
\[= 2[A(Z)R'(X, Y, V, U) - A(U)R'(X, Y, V, Z)]\]

After using $g(X, T) = A(X)$ and $Ric(X, Y) = (n-1)g(X, Y)$

\[= 2[A(Z)R'(X, Y, V, U) - A(U)R'(X, Y, V, Z)]\]

Since $A(X)$ and $A(Y)$ are non-zero, $W_3' = 0$ from the definition of $W_3$ symmetric. Also the coefficients of $A(X)$ cancelled out since $R'$ is skew-symmetric with respect to the last two variables. Same to the coefficients of $A(U)$

And thus leaves us with
\[2[g(U, V)R'(X, Y, Z, T) - g(Z, V)R'(X, Y, U, T)] = 0\]

Since $g(U, V) \neq g(Z, V) \neq 0$. For arbitrary vectors $U, V, Z, ...$

Implying that if $W_3$ is symmetric i.e.
\[\nabla_X W_3(X, Y, Z) = W_3'(X, Y, Z, U) = 0\]
\[\Rightarrow R'(X, Y, Z, T) = 0\]

Hence the theorem
If

$$R(X,Y)W_3(U,V)Z = 0 \Rightarrow g(R(X,Y)W_3, T) = R'(X,Y, W_3, T) = g(X, T)g(Y, W_3) -$$

$$g(Y, T)g(X, W_3) = A(X)W_3'(Z, U, V, X) - A(Y)W_3'(Z, U, V, X) = 0$$

Since $A(X)$ and $A(Y)$ are non-zero $\Rightarrow W_3' = 0$ i.e.

$$W_3'(Z, U, V, Y) = W_3'(Z, U, V, X) = 0$$

$$\nabla_U W_3(X, Y, Z) = W_3'(X, Y, Z, U) = 0$$

Hence the theorem.

V. Discussion

The $W_3'$-curvature tensor is symmetrical in $Z$ and $U$ and satisfies Cyclic property with fixed $X$ [5]. The Rainich conditions for existence of non-null electro variance can be obtained by the contracted part of this tensor [6].

Reference