International Journal of Innovation in Science and Mathematics
Volume 6, Issue 3, ISSN (Online): 2347–9051

A Study of W_3 - Symmetric K-Contact Riemannian Manifold

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Date of publication (dd/mm/yyyy): 17/05/2018

Abstract **– In this paper the geometric properties of W_3 - curvature tensor are studied in K-contact Riemannian manifold.**

Keywords **– W_3 - Curvature Tensor, K-Contact Riemannian Manifold, Semi Symmetric, Symmetric and W_3 - Flat.**

I. 2010 Mathematics Subject Classification: 53C15, 53C40

(1.0) Preliminaries:

Let M_n be an $n(=2m+1)-$ dimensional contact Riemannian manifold with the structure tensors (Φ, T, A, g).

Then the following formulas holds:

(1.1) $\phi^2 X = -X + A(X)T, \qquad A(T) = 1$

$$\phi T = 0, \quad g(X,T) = A(X) \quad g(\phi X, \phi Y) = g(X,Y) - A(X)A(Y) \tag{1.2}$$

$$F(X,Y) = -g(\emptyset X, Y) = +g(X, \emptyset Y) = (\nabla_X A)(Y) = -(\nabla_Y A)(X) \tag{1.3}$$

$dA(X,Y) = g(X, \phi Y)$ [see [1]] for any vector fields X and Y in M. If we defined an operator h by $\bar{h} = \frac{1}{2} L_T \phi$ where h is the lie derivative, then any contact Riemannian manifold satisfies the condition that h and $\phi\bar{h}$ are symmetric operators, h ant-commutes with ϕ (i.e. $\phi\bar{h} + \bar{h}\phi$=0), $Ao\bar{h} = 0$ see[2] and [3] (1.4) $\bar{h}T = 0$ and e ($\nabla_X T = -\phi X - \phi\bar{h}$X), A contact Riemannian manifold is said to be K-contact if

(1.5) If $\nabla_X T = -\phi X$ also in K-contact we have

(1.6) $(\nabla_Y F)(Z, X) = R(Z, X, Y, T)$

(1.7)
$$(\nabla_Z F)(\phi X, \phi Y) + (\nabla_Z F)(X, Y) - A(Y)A(\nabla_Z \phi X) + A(X)A(\nabla_Z \phi Y) = 0,$$

(1.8)
$$R(X,Y,Z,T) + R(\phi X, \phi Y, \phi Z, \phi T) = A(Y)A(\nabla_Z \phi X - A(X)A(\nabla_Z \phi Y)),$$

(1.9) $A(\nabla_Y \emptyset X) = A(X)A(Y) - g(X,Y)$

(1.10) $S(T,T) = Ric(T,T) = n-1,$

where R is the Riemannian (0,4) curvature tensors S = Ric (.,.) is the Ricci tensor and $F(X,Y) = g(\phi X, Y)$.

(1.11) $W_3 -$ tensor in K-contact Riemannian Mishra and Pokhariyal [4] gave the definition of W_3-tensor as

(1.12)
$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - Ric(X,Z)Y]$$

where Q is the symmetric endomorphism of tangent space at a point to the Ricci tensor or

$$W_3'(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z)]$$

II. W_3-Curvature Tensor in K-Contact Riemannian Manifold

A K-contact Riemannian manifold is said to be W_3-flat if $W_3(X,Y)Z = 0$.

(2.1) Theorem:

A $W_3 -$ flat K-contact Riemannian Manifold is a Space of negative scalar curvature, that is: $r = -n(n+1)$

Proof : Since

$$W_3'(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z)]$$

Put $W_3' = 0$

$$\Rightarrow 0 = R(X,Y,Z,U) + \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z)]$$

$$\Rightarrow R(X,Y,Z,U) = \frac{1}{n-1}[g(Y,U)Ric(X,Z) - g(Y,Z)Ric(X,U)]$$

Then using the definition of $Ric(X,Y) = (n-1)g(X,Y)$ in the above equation

we get

$$R(X,Y,Z,U) = \frac{1}{n-1}[(n-1)g(Y,U)g(X,Z) - (n-1)g(Y,Z)g(X,U)]$$
$$= [g(Y,U)g(X,Z) - g(Y,Z)g(X,U)]$$
$$= -[-g(Y,U)g(X,Z) + g(Y,Z)g(X,U)]$$
$$= -[g(Y,Z)g(X,U) - g(Y,U)g(X,Z)],$$

which on contraction we get the result hence the theorem.

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III. W_3 - Symmetric K-Contact Riemannian Manifold

A K-contact Riemannian manifold is said to be symmetric if.

(3.1) $\nabla_U W_3(X,Y,Z) = W_3{}'(X,Y,Z,U) = 0.$

Theorem 3:

A W_3-symmetric and W_3-flat K-contact manifold is a flat manifold i.e. zero curvature.

Proof:

From the symmetric property it follow

(3.1.1)

$$R(X,Y,W_3(Z,U,V)) - W_3(R(X,Y,Z),U,V) - W_3(Z,R(X,Y,U),V) - W_3(Z,U,R(X,Y,V)) = 0$$

We expand the above equation

(3.1.2)

$$R(X,Y,W_3(Z,U,V)) = g(Y,W_3(Z,U,V))X - g(X,W_3(Z,U,V))Y = W_3{}'(Y,Z,U,V)X - W_3{}'(X,Z,U,V)Y$$

$$\Rightarrow R'(X,Y,W_3(Z,U,V),T) = W_3{}'(Y,Z,U,V)A(X) - W_3{}'(X,Z,U,V)A(Y)$$

(3.1.3) $W_3{}'(R(X,Y,Z),U,V))$

$$= R(R(X,Y,Z),U,V,T) + \frac{1}{n-1}[g(U,V)Ric(R(X,Y,Z),T) - g(U,T)Ric(R(X,Y,Z),V)]$$

$$= R'(R(X,Y,Z),U,V,T) + \frac{n-1}{n-1}[g(U,V)R'(X,Y,Z,T) - A(U)R'(X,Y,Z,V)]$$

$$= g(U,V)R'(X,Y,Z,T) - A(U)R'(X,Y,Z,V) + g(U,V)R'(X,Y,Z,V) - A(U)R'(X,Y,Z,V)$$

$$= 2\{g(U,V)R'(X,Y,Z,T) - A(U)R'(X,Y,Z,V)$$

(3.1.4)

$$W_3{}'(Z,R(X,Y,U),V,T)) = R'(Z,R(X,Y,U),V,T) + \frac{1}{n-1}[g(R(X,Y,U),V)Ric(Z,T) - g(R(X,Y,U),T)Ric(Z,U)]$$

$$= A(Z)R'(X,Y,U,V) - R'(X,Y,U,T)g(Z,V) + \frac{n-1}{n-1}[A(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,T)]$$

$$= 2\{A(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,T)\}$$

After using $g(X,T) = A(X)$ and $Ric(X,Y) = (n-1)g(X,Y)$

(3.1.5)

$$W_3{}'(Z,U,R(X,Y,V),T)) = R'(Z,U,R(X,Y,V),T) + \frac{1}{n-1}[g(U,R(X,Y,V))Ric(Z,T) - g(U,T)Ric(R(X,Y,V),Z]$$

$$= R'(Z,U,R(X,Y,V),T) + \frac{n-1}{n-1}[A(Z)R'(X,Y,V,U) - A(U)R'(X,Y,V,Z)]$$

$$= 2\{A(Z)R'(X,Y,V,U) - A(U)R'(X,Y,V,Z)\},$$

After using $g(X,T) = A(X)$ and $Ric(X,Y) = (n-1)g(X,Y)$.

Putting together (3.1.2), (3.1.3), (3.1.4) and (3.1.5) we get

$$W_3{}'(Y,Z,U,V)A(X)) - W_3{}'(X,Z,U,V)A(Y)) + 2\{g(U,V)R'(X,Y,Z,T) - A(U)R'(X,Y,Z,V)\} + 2\{A(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,T)\} + 2\{A(Z)R'(X,Y,V,U) - A(U)R'(X,Y,V,Z) = 0$$

Since $A(X)$ and $A(Y)$ are non-zero, $W_3{}' = 0$ from the definition of W_3 symmetric. Also the coefficients of $A(X)$ cancelled out since R' is skew-symmetric with respect to the last two variables. Same to the coefficients of $A(U)$.

And thus leaves us with

$$2\{g(U,V)R'(X,Y,Z,T) - g(Z,V)R'(X,Y,U,T)\} = 0$$

Since $g(U,V) \neq g(Z,V) \neq 0$. For arbitrary vectors $U, V, Z, \ldots$

Implying that if W_3 is symmetric i.e.

$$\nabla_X W_3(X,Y,Z) = W_3{}'(X,Y,Z,U) = 0$$
$$\Rightarrow R'(X,Y,Z,T) = 0$$

Hence the theorem

IV. W_3 - Semi Symmetric K-Contact Riemannian Manifold

A K-contact Riemannian manifold is said to be a W_3 Semi symmetric if

$$R(X,Y)W_3(U,V)Z = 0.$$

Theorem 4:

A W_3-Semi symmetric K-contact Riemannian manifold is a W_3- symmetric K-contact manifold.

Proof:

If

$R(X,Y)W_3(U,V)Z = 0 \Rightarrow g(R(X,Y)W_3,T) = R'(X,Y,W_3,T) = g(X,T)g(Y,W_3) - g(Y,T)g(X,W_3) = A(X)W_3'(Z,U,V,Y) - A(Y)W_3'(Z,U,V,X) = 0$

Since $A(X)$ and $A(Y)$ are non-zero $\Longrightarrow W_3' = 0$ i.e. $W_3'(Z,U,V,Y) = W_3'(Z,U,V,X) = 0$ but $\nabla_U W_3(X,Y,Z) = W_3'(X,Y,Z,U) = 0$

Hence the theorem.

V. Discussion

The W_3-curvature tensor is symmetrical in Z and U and satisfies Cyclic property with fixed X [5]. The Rainich conditions for existence of non- null electro variance can be obtained by the contracted part of this tensor [6].

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