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Application of Vandermonde Determinant in Determinant Calculation

Rui Chen[1*], Liang Fang[2], Chuan Tian[3] and Guocai Liu[4]
[1,2,4]College of Mathematics and Statistics, Taishan University, Tai'an, Shandong, China.
[3]Taian Technician Institute, Tai'an, Shandong, China.
[*]Corresponding author email id: chenruimengting@163.com

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***Abstract* – Vandermonde determinant is a kind of important determinant. This paper discusses the calculation of Vandermonde determinant and how to convert some special determinants into Vandermonde determinant for calculation, so as to reduce the amount of calculation and improve the efficiency of calculation.**

***Keywords* – Vandermonde Determinant, Determinant Calculation, Determinant Property.**

I. INTRODUCTION

A. *Relationship between Vandermonde Determinant and Determinant Calculation*

Determinant as a function can be regarded as the extension of the concept of directed area or volume in general Euclidean space, the calculation of determinant is an important content in linear algebra. Vandermonde determinant is a special kind of determinant. It has a unique standard form and concise calculation results [1]. There are many methods for calculating determinants, such as the first-order transformation method, the method of reducing or increasing order, the method of block matrix, the method of recursion, the method of mathematical induction, etc[2-8]. Among these basic thought methods, the method of reducing order, that is, the method of transforming high-order determinants into low-order determinants, is one of the core thought methods in linear algebra to deal with matrix and determinant-related problems. For example, Laplace theorem is the basic method for finding determinants [3], which fully embodies linear algebra.

B. *Definition and Value of Vandermonde Determinant*

$D=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \cdots & \cdots & \ddots & \cdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$. The determinant is called the Vandermonde determinant, and its value is $D=\prod_{1\le j<i\le n}(a_i-a_j)$ [4].

In the traditional teaching materials, we use mathematical induction to prove the Vandermonde determinant, and reduce the Vandermonde determinant by elementary row transformation to get its value. In fact, using the method of determinant column transformation will make the problem solving more flexible.

II. APPLICATION OF VANDERMONDE DETERMINANT IN DETERMINANT CALCULATION

A. *Directly using the Result of Vandermonde Determinant*

Example 1. Calculate the value of this determinant: $D=\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 2^2 & 3^2 & 4^2 & 5^2 & 6^2 \\ 2^3 & 3^3 & 4^3 & 5^3 & 6^3 \\ 2^4 & 3^4 & 4^4 & 5^4 & 6^4 \end{vmatrix}$.

According to the definition of Vandermonde determinant, this determinant is vandermond determinant, and its value is as follows.

$$D=\prod_{2\le j<i\le 6}(i-j)=(6-2)(6-3)(6-4)(6-5)\begin{matrix}(5-2)(5-3)(5-4)\\(4-2)(4-3)\\(3-2)\end{matrix}\begin{matrix}=4\times3\times2\times1\times3\times2\times1\times2\times1\times1\\=288.\end{matrix}$$

B. *Using the Properties of Determinant*

a. *Extraction Common Factor Method*

Example 2. Calculate the value of this determinant: $D=\begin{vmatrix}1 & 2 & 3 & \cdots & n\\ 1 & 2^2 & 3^2 & \cdots & n^2\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ 1 & 2^n & 3^n & \cdots & n^n\end{vmatrix}$.

Each column of this determinant has a common factor. After the common factor is proposed from each column, the determinant becomes a Vandermonde determinant.

$$D=n!\begin{vmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & 2 & 3 & \cdots & n\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1}\end{vmatrix}=n!(n-1)!(n-2)!\cdots 2!1!.$$

b. *Row Transformation Method*

Example 3. Calculate the value of this determinant:

$$D=\begin{vmatrix}1 & 1 & 1 & 1\\ 1+\cos A_1 & 1+\cos A_2 & 1+\cos A_3 & 1+\cos A_4\\ \cos A_1+\cos^2 A_1 & \cos A_2+\cos^2 A_2 & \cos A_3+\cos^2 A_3 & \cos A_4+\cos^2 A_4\\ \cos^2 A_1+\cos^3 A_1 & \cos^2 A_2+\cos^3 A_2 & \cos^2 A_3+\cos^3 A_3 & \cos^2 A_4+\cos^3 A_4\end{vmatrix}.$$

Multiply the previous row of the determinant by -1 to the next row, and then multiply the result by -1 to the next row to obtain a Vandermonde determinant [5].

$$D=\begin{vmatrix}1 & 1 & 1 & 1\\ \cos A_1 & \cos A_2 & \cos A_3 & \cos A_4\\ \cos^2 A_1 & \cos^2 A_2 & \cos^2 A_3 & \cos^2 A_4\\ \cos^3 A_1 & \cos^3 A_2 & \cos^3 A_3 & \cos^3 A_4\end{vmatrix}=(\cos A_4-\cos A_1)(\cos A_3-\cos A_1)(\cos A_2-\cos A_1)$$
$$(\cos A_3-\cos A_1)(\cos A_3-\cos A_2)(\cos A_2-\cos A_1).$$

c. *Ascending Order Method*

Example 4. Calculate the value of this determinant: $D_n=\begin{vmatrix}1 & b_1 & b_1^2 & \cdots & b_1^{n-2} & b_1^n\\ 1 & b_2 & b_2^2 & \cdots & b_2^{n-2} & b_2^n\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & b_{n-1} & b_{n-1}^2 & \cdots & b_{n-1}^{n-2} & b_{n-1}^n\\ 1 & b_n & b_n^2 & \cdots & b_n^{n-2} & b_n^n\end{vmatrix}$.

According to the characteristics of determinant, add a row and a column by adding edges.

$$D_{n+1}=\begin{vmatrix}1 & b_1 & b_1^2 & \cdots & b_1^{n-2} & b_1^{n-1} & b_1^n\\ 1 & b_2 & b_2^2 & \cdots & b_2^{n-2} & b_2^{n-1} & b_2^n\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 1 & b_{n-1} & b_{n-1}^2 & \cdots & b_{n-1}^{n-2} & b_{n-1}^{n-1} & b_{n-1}^n\\ 1 & b_n & b_n^2 & \cdots & b_n^{n-2} & b_n^{n-1} & b_n^n\\ 1 & a & a^2 & \cdots & a^{n-2} & a^{n-1} & a^n\end{vmatrix}.$$

Expand the determinant above by the last row, $D_{n+1}=A_{n+1,1}\bullet 1+A_{n+1,2}\bullet a+A_{n+1,3}\bullet a^2+\cdots+A_{n+1,n}\bullet a^{n-1}+A_{n+1,n+1}\bullet a^n$,

Coefficient of a^{n-1} is $A_{n+1,n}=(-1)^{n+1+n}\bullet D_n=-D_n.$

D_{n+1} is the transposed vandermond determinant, the result of the vandermond determinant, $D_{n+1}=(a-b_1)(a-b_2)\cdots(a-b_n)\prod_{1\le j<i\le n}(b_i-b_j)$.

Coefficient of a^{n-1} is $-(b_1+b_2+\cdots b_n)\prod_{1\le j<i\le n}(b_i-b_j)$.

Comparison coefficient, $D_n=(b_1+b_2+\cdots b_n)\prod_{1\le j<i\le n}(b_i-b_j)$.

d. *Split Item Method*

Example 5. Calculate the value of this determinant: $D_{n+1}=\begin{vmatrix}1 & -1 & -1 & \cdots & -1\\ 1 & b_1 & b_1^2 & \cdots & b_1^n\\ 1 & b_2 & b_2^2 & \cdots & b_2^n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & b_n & b_n^2 & \cdots & b_n^n\end{vmatrix}$.

Break the first line into the sum of two terms, $D_{n+1}=\begin{vmatrix}2 & 0 & 0 & \cdots & 0\\ 1 & b_1 & b_1^2 & \cdots & b_1^n\\ 1 & b_2 & b_2^2 & \cdots & b_2^n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & b_n & b_n^2 & \cdots & b_n^n\end{vmatrix}-\begin{vmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & b_1 & b_1^2 & \cdots & b_1^n\\ 1 & b_2 & b_2^2 & \cdots & b_2^n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & b_n & b_n^2 & \cdots & b_n^n\end{vmatrix}$

$$=2b_1b_2\cdots b_n\prod_{1\le j<i\le n}(b_i-b_j)-\prod_{i=1}^{n}(b_i-1)\prod_{1\le j<i\le n}(b_i-b_j)$$

$$=\left[2b_1b_2\cdots b_n-\prod_{i=1}^{n}(b_i-1)\right]\prod_{1\le j<i\le n}(b_i-b_j).$$

III. Conclusion

Starting from the calculation results of Vandermonde determinant and combined with the calculation properties of determinant, this paper discusses some special the determinant similar to the Vandermonde determinant is transformed into the Vandermonde determinant for calculation, which finally turns complexity into simplicity, so as to achieve twice the result with half the effort.

Vandermonde determinant is a famous determinant in linear algebra. It has unique structure and beautiful form. Moreover, it has become a famous determinant because of its wide application. As long as you are familiar with the applicable form and application skills of Vandermonde determinant, you can well apply

Vandermonde determinant to calculate relevant determinants. Van der Monde determinant is widely used not only in mathematics, but also in other fields.

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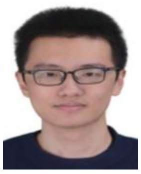
AUTHOR'S PROFILE

First Author
Rui Chen, is a associate professor at Taishan University. She obtained her master's degree from Shandong University in December, 2009. Her research interests are in the areas of application of probability theory, and applied statistics in recent years.

Second Author
Liang Fang, is a professor at Taishan University. He obtained his PhD from Shanghai Jiaotong University in June, 2010. His research interests are in the areas of cone optimization, and complementarity problems.



Third Author
Chuan Tian, is a teacher of Taian Technician Institute, graduated from the Department of mathematics of Shandong University in June 2020, majoring in mathematics and applied mathematics, obtained a bachelor's degree, and is committed to basic mathematics and mathematics teaching research.

Fourth Author
Guocai Liu, is an associate professor of Taishan University. He graduated from Northeast Normal University in July 1993, and is committed to the teaching and research of basic mathematics in Colleges.