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Fixed Point Theorems for Generalized Contractions on S-Metric Spaces

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Abstract: **Introducing S-metric spaces, Shaban Sedghi, Nabi Shobe and Abdelkrim Aliouche [4] have proved a result similar to Banach's contraction principle. We generalize this theorem by using λ- contraction on an S-metric space in this paper.**

Keywords: **S-Metric Space, F-Orbitally Complete Spaces, λ- Contraction.**

I. Introduction

The concept of a metric on a non empty set X has been generalized in many ways in recent years. Infact, Shaban Sedghi ,Nabi Shobe and Haiyan Zhou[3] have introduced D*-metric; Zead Mustafa and Brailey sims[5] initiated G-metric while Shaban Sedghi, Nabi Shobe and Abdalkrim Aliouche [4] considered S-metric . In ([4] Remark 1.3 and Remark 2.2) it is claimed that every G-metric is a D*-metric and that every D*-metric is an S-metric.This first claim has been disproved by the authors in a recent paper[2]. Infact, it has been proved by examples ,that D* -metric do not imply those of a S-metric and vice versa. Further, it was proved that G-metric and S-metric on a non empty set X are also independent in the above sense.

In this paper we consider S-metric spaces and prove fixed point theorems for selfmaps of such spaces. Our result generalizes a fixed point theorem proved in [4].

II. S-Metric Spaces and λ- Contractions

In this section we present some preliminary results needed for our purpose. We begin with

2.1. ***Definition*** ([4], Definition 2.1). Let X be a non empty set.

An *S-metric* on X is a function S: $X^3 \to [0,\infty)$ that satisfies the conditions given below for $x,y,z,w \in X$:

(i) $S(x,y,z) \geq 0$

(ii) $S(x,y,z) = 0$ if and only if $x = y = z$ and

(iii) $S(x,y,z) \leq S(x,x,w) + S(y,y,w) + S(z,z,w)$

The pair $(X,S)$is called an *S-metric space.*

If $(X,S)$is an S-metric space it is shown in ([4], Lemma 2.5) that

(2.2) $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$ and as a consequence of (iii) of Definition 2.1 and (2.2) we have

(2.3) $S(x,x,y) \leq 2.S(x,y,z) + S(y,y,z)$ for $x,y,z \in X$

A Sequence $\{x_n\}$ in (X,S) is said to

(i) *converge* to x if to each $\varepsilon > 0$ there is a natural number n_0 such that $S(x_{n,}x_{n,}x) < \varepsilon$ for all $n \geq n_0$ and

(ii) be a *Cauchy Sequence* if to each $\varepsilon > 0$ there is a natural number n_0 such that $S(x_{m,}x_{m,}x_n) < \varepsilon$ for all $m \geq n_0, n \geq n_0$.

It is shown in ([4], Lemma 2.10 and Lemma 2.11) that in an S-metric space (X,S) if $\{x_n\}$ converges to x then x is unique and that $\{x_n\}$is a Cauchy Sequence.

An S-metric space is said to be *complete* if every Cauchy Sequence in it converges to a point in X.

It is easy to prove :

(2.4) If $\{x_n\}$ and $\{y_n\}$ are sequences in an S-metric space converging respectively to x and y

then $\lim_{n\to\infty} S(x_{n,}x_{n,}y_n) = S(x,x,y)$

(2.5) ***Definition*** ([4], Definitio n 2.13). A Selfmap f of (X,S) is said to be a *contraction* if there is a constant L with $0 \leq L < 1$ such that

(2.5.1) $S(fx,fx,fy) \leq L.S(x,x,y)$ for all $x,y \in X$. As usual , for $f: X \to X$ its n^{th} *iterate*, denoted by f^n is defined by $f^0(x) = x$ and $f^n(x) = f(f^{n-1}x)$ for $n \geq 1$.

The following result similar to Banach's contraction principle has been proved in ([4],Theorem 3.1)

2.6. ***Theorem:*** Let (X,S) be a complete S-metric space and $f: X \to X$ be a contraction. Then f has a unique fixed point $x \in X$. Also for any $x \in X$, $\lim_{n\to\infty} f^n(x) = u$ and $S(f^n x, f^n x, u) \leq 2.\frac{L^n}{1-L}.S(x,x,fu)$.

The notion of contraction defined in metric space theory is generalized by Lj.B.Ciric[1]. Analogously we define generalized contractions in S-metric spaces as follows:

2.7. ***Definition****:* A selfmap f on an S-metric space (X,S) is said to be a λ- *contraction* if there exist non-negative functions q,r,s and t (all defined on X×X) such that

$\lambda = \sup_{(x,y)\in X\times X} \{q(x,y) + r(x,y) + s(x,y) + 3t(x,y)\} < 1$ and for all $x,y \in X$ the inequality

(2.7.1) $S(fx,fx,fy) \leq q.S(x,x,y) + r.S(x,x,fx) + s.S(y,y,fy) + t\{S(x,x,fy) + S(y,y,fx)\}$ holds.

2.8. ***Remark:*** If $q(x,y) =$ L (a constant with $0 \leq L < 1$) and $r(x,y) = s(x,y) = t(x,y) = 0$ for all $x,y \in X \times X$ then $\lambda = L$ so that the inequality (2.7.1) reduces to (2.5.1). Thus a contraction is a λ- contraction.

We give below an example of a λ- contraction which is not a contraction.

2.9. ***Example :***

Suppose $X = [0,2]$. Define $f: X \to X$ by $f(x) = \begin{cases} \frac{x}{9}, if\ x \in [0,1) \\ \frac{x}{10}, if\ x \in [1,2] \end{cases}$

Let $S(x,y,z) = \frac{1}{2}(|x-y|,|y-z|,|z-x|)$ for $x,y,z \in X$ so that $(X,S)$is a S-metric space.

Let $q(x,y) = \frac{1}{10}, r(x,y) = \frac{1}{8}, s(x,y) = \frac{1}{4}$ and $t(x,y) = \frac{1}{6}$ for all $x,y \in X \times X$. Then

$$\lambda = \sup_{(x,y)\in X\times X} \{q(x,y) + r(x,y) + s(x,y) + 3t(x,y)\}$$
$$= \frac{1}{10} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} = \frac{4+5+10+20}{40} = \frac{39}{40} < 1$$

we claim that

(2.9.1) $S(fx,fx,fy) \le R(x,y)$ for all $x,y \in X$,where

$$R(x,y) = q.S(x,x,y) + r.S(x,x,fx) + s.S(y,y,fy) + t\{S(x,x,fy) + S(y,y,fx)\}$$

Case(i) : Suppose $x,y \in [0,1]$

Then $S(fx,fx,fy) = |fx - fy| = \left|\frac{x}{9} - \frac{y}{9}\right| = \frac{1}{9}|x-y|$ while

$$R(x,y) = \frac{1}{10}|x-y| + \frac{1}{8}\left|x - \frac{x}{9}\right| + \frac{1}{4}\left|y - \frac{y}{9}\right| + \frac{1}{2}\left\{\left|x - \frac{y}{9}\right| + \left|y - \frac{x}{9}\right|\right\}$$

proving (2.9.1) in this case.

Case(ii) **:** Suppose $x \in [0,1]$ and $y \in (1,2]$ so that $S(fx,fx,fy) = \left|\frac{x}{9} - \frac{y}{10}\right|$ while

$$R(x,y) = \frac{1}{10}|x-y| + \frac{1}{8}\left|x - \frac{x}{9}\right| + \frac{1}{4}\left|y - \frac{y}{10}\right| + \frac{1}{6}\left\{\left|x - \frac{y}{10}\right| + \left|y - \frac{x}{9}\right|\right\} > \frac{x}{9} + \frac{9y}{40} > \frac{x}{9} + \frac{y}{10} > \left|\frac{x}{9} - \frac{y}{10}\right|$$

proving (2.9.1) in this case.

In the other two cases of (iii) $x,y \in (1,2]$; and (iv) $x \in (1,2]$ and $y \in [0,1]$ one can prove (2.9.1)

Thus f is a λ- contraction. But this is not a contraction as given in Definition 2.5. In fact, if $x_0 = \frac{999}{1000}$ and $y_0 = \frac{1001}{1000}$ then $x_0, y_0 \in X$ and $S(fx_0, fx_0, fy_0) = \left|\frac{111}{1000} - \frac{1001}{10000}\right| = \frac{109}{10000}$ while $S(x_0,x_0,y_0) = \left|\frac{999}{1000} - \frac{1001}{1000}\right| = \frac{2}{1000}$ showing $S(fx_0,fx_0,fy_0) > S(x_0,x_0,y_0)$

2.10 ***Definition:*** If $f: X \to X$ is a map then for any x $\in X$, *the orbit of* x *relative to* f, denoted by $O_f(x,\infty)$is defined by

$O_f(x;\infty) = \{x, fx, f^2x, \ldots f^n x, \ldots\}$,

where $f^n$is the n[th] iterate of f.

2.11. ***Definition:*** If f is a selfmap of an S-metric space (X,S)then X is said to be f*-orbitally complete* if for some $x_0 \in X$, every Cauchy sequence in $O_f(x_0;\infty)$ converges to a point in X.

Clearly if (X,S) is complete then it is f-orbitally complete for every self map f of X.

III. Main Results

The first theorem we prove here is the following:

*3.1.****Theorem:*** Suppose f is a selfmap of an S-metric space (X,S) which is f*-orbitally complete*. If f is a λ-contraction then it has a unique fixed point $u \in X$.

Also $\lim_{n\to\infty} f^n(x_0) = u$ and $S(f^n x_0, f^n x_0, u) \le 2.\frac{\lambda^n}{1-\lambda} S(fx_0, fx_0, x_0)$,

where $x_0 \in X$ is such that every Cauchy sequence in $O_f(x_0;\infty)$ converges to a point in X.

Proof: Let $x_0 \in X$ be such that every Cauchy sequence in its orbit $O_f(x_0;\infty)$ converges to a point in X.

Writing $x_n = f^{n-1}x_0$ for n=1,2,3, . . .

we find $x_1 = x_0, x_2 = fx_0, \ldots\ldots, x_n = fx_{n-1}$ for $n \ge 1$.

Now for any $n \ge 1$, note that, by (2.7.1), (2.2) and (2.3) we have

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) &= S(fx_n, fx_n, fx_{n-1}) \\ &\le q.S(x_n, x_n, x_{n-1}) + r.S(x_n, x_n, fx_n) \\ &\quad + s.S(x_{n-1}, x_{n-1}, fx_{n-1}) \\ &\quad + t\{S(x_n, x_n, fx_{n-1}) \\ &\quad + S(x_{n-1}, x_{n-1}, fx_n)\} \end{aligned}$$

$= q.S(x_n,x_n,x_{n-1}) + r.S(x_n,x_n,x_{n+1}) + s.S(x_{n-1},x_{n-1},x_n) t\{S(x_n,x_n,x_n) S(x_{n-1},x_{n-1},x_{n+1})\} \le (q+s).S(x_n,x_n,x_{n-1}) + r.S(x_n,x_n,x_{n+1}) + t\{2S(x_{n-1},x_{n-1},x_n) + S(x_{n+1},x_{n+1},x_n)\}$

$= (q+s+2t).S(x_n, x_n, x_{n-1}) + (r+t).S(x_{n+1},x_{n+1},x_n)$

so that

$$(1-r-t).S(x_{n+1},x_{n+1},x_n) \le (q+s+2t) S(x_n,x_n,x_{n-1})$$

That is ,for every $n \ge 1$,

(3.1.1)

$$S(x_{n+1},x_{n+1},x_n) \le \frac{q+s+2t}{1-r-t}.S(x_n,x_n,x_{n-1}) \le \lambda.S(x_n,x_n,x_{n-1})$$

By repeated application of (3.1.1) we get

(3.1.2)
$$\begin{aligned} S(x_{n+1},x_{n+1},x_n) &\le \lambda.S(x_n,x_n,x_{n-1}) \\ &\le \lambda^2.S(x_{n-1},x_{n-1},x_{n-2}) \\ &\le \ldots \\ &\quad \ldots \\ &\le \lambda^n S(x_1,x_1,x_0) \end{aligned}$$

Therefore for any integer p ≥ 0 ,we have by (2.3) and (3.1.2) that

(3.1.3)
$$\begin{aligned} S(x_{n+p},x_{n+p},x_n) &\le 2.S(x_{n+1},x_{n+1},x_n) + S(x_{n+1},x_{n+1},x_{n+p}) \\ &\le 2.S(x_{n+1},x_{n+1},x_n) + 2.S(x_{n+1},x_{n+1},x_{n+2}) + S(x_{n+2},x_{n+2},x_{n+p}) \\ &\le \ . \ . \ . \\ &\le 2.\lambda^n.S(x_1,x_1,x_0) + 2\lambda^{n+1}.S(x_1,x_1,x_0) + \ldots \\ &\le 2\lambda^n.S(x_1,x_1,x_0)(1+\lambda+\lambda^2+\ldots) \\ &= \frac{2\lambda^n}{1-\lambda} S(x_1,x_1,x_0) \end{aligned}$$

Now since $\lambda^n \to 0$ as n $\to \infty$, it follows from (3.1.3) that $\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_{n+p}) = 0$ showing that $\{x_n\}$ is a Cauchy sequence and hence converges to some point $u \in X$.

That is,

(3.1.4) $u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0$

To prove that u is a fixed point of f, we first observe that

$$S(x_{n+1},x_{n+1},fu) = S(fx_n,fx_n,fu)$$
$$\leq q.S(x_n,x_n,u) + r.S(x_n,x_n,x_{n+1}) + s.S(u,u,fu) + t\{S(x_n,x_n,fu) + S(u,u,x_{n+1})\}$$
$$\leq q.S(x_n,x_n,u) + r.S(x_n,x_n,x_{n+1}) + s\{2.S(u,u,x_n) + S(x_n,x_n,fu)\} + t\{S(x_n,x_n,fu) + S(u,u,x_{n+1})\}$$

$$= (q+2s)\, S(x_n,x_n,u) + r.s(x_n,x_n,x_{n+1}) + (s+t)S(x_n,x_n,fu) + t.S(u,u,x_{n+1})$$
$$\leq (q+2s).S(x_n,x_n,u) + r.S(x_n,x_n,x_{n+1}) + (s+t)\{2.S(x_n,x_n,x_{n+1}) + S(x_{n+1},x_{n+1},fu)\} + t.S(u,u,x_{n+1})$$

from which we find

$$0 \leq (1-s-t)S(x_{n+1},x_{n+1},fu)$$
$$\leq (q+2s).S(x_n,x_n,u) + r.S(x_n,x_n,x_{n+1}) + 2(s+t).S(x_n,x_n,x_{n+1}) + t.S(u,u,x_{n+1})$$

Now letting n to ∞ on either side and using (3.1.4) we get $\lim_{n\to\infty} S(x_{n+1},x_{n+1},fu) = 0$ showing

(3.1.5) $fu = \lim_{n\to\infty} x_n$

Hence by (3.1.4) and (3.1.5) we get $fu = u$

For the uniqueness of u, let $v \in X$ be such that $fv = v$

Then $S(u,u,v) = S(fu,fu,fv)$
$$\leq q.S(u,u,v) + r.S(u,u,fu) + s.S(v,v,fv) + t\{S(u,u,fv) + S(v,v,fu)\}$$
$=q.S(u,u,v) + 2t\, S(u,u,v)$=$(q +$ *2t.Su,u,v<λ.S(u,u,v)*

which implies $S(u,u,v) = 0$ since $0 < \lambda < 1$, proving $u = v$.

3.2. ***Remark:*** Since every complete $S-$metric space (X,S) is $f-$orbitally complete for every self map f of X and since every contraction is a λ-contraction, Theorem 2.6 follows from Theorem 3.1.

3.3. ***Theorem****:* Suppose f is a selfmap of a $S-$metric space (X,S) which is f-orbitally complete. If there is a positive integer k such that $f^k$is a λ-contraction then f has a unique fixed point $u \in X$.

Proof: For any self map f of X , first note that $O_{f^k}(x;\infty) \subseteq O_f(x;\infty)$for any $x \in X$ and integer $k \geq 0$. Therefore if X is f- orbitally complete $S-$metric space then it is f^k- orbitally complete also. Now, by the theorem, $f^k$is a λ-contraction on the $S-$metric space (X,S) which is f^k- orbitally complete .Therefore, by the Theorem 3.1, f^k has a unique fixed point $u \in X$. That is, $f^k(u) = u$.

Now $f^k(fu) = f^{k+1}(u) = f(f^k u) = f(u) = fu$ shows that fu is also a fixed point of f^k.Therefore by the uniqueness of fixed point for f^k, it follows that $fu = u$. That is, $u \in X$is a fixed point of f.

To prove the uniqueness of fixed point for f, assume $v \in X$ is such that $fv = v$.Then it is clear that $f^k v = v$ for any integer $k \geq 0$ showing v is also a fixed point of f^k.By the uniqueness of the fixed point for f^k it follows $u = v$.

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