Imbedding Inequalities and Poincare Inequalities of Weakly A-harmonic Tensors

Chunxia Gao
I. College of Electronic and Information Engineering, Hebei University, Baoding 071002, China

Yuxia Tong
2. College of Science, Hebei United University, Tangshan 063009, China

Abstract — The Imbedding Inequalities and Poincare Inequalities of Weakly A-harmonic tensors have been proved.

Keywords — Weakly A-Harmonic Tensor, Differential Form, Imbedding Inequality, Poincare Inequalities.

I. INTRODUCTION

Given \( \Omega \in L^r(\Omega, \Lambda') \) and \( f \in L^{r-\beta}(\Omega, \Lambda') \) where \( r \geq 1 \) \( \{1, p - 1\} \), we consider the non homogeneous equation for differential forms

\[
d'A(x, g + du) = d'f \quad \text{for} \quad u \in W^{1, r}(\Omega, \Lambda') \tag{1.1}
\]

Where \( A: \Omega \times \Lambda' \to \Lambda^{r, p} \) satisfies the conditions

(H1) \[ |A(x, \xi) - A(x, \zeta)| \leq \beta |\xi - \zeta| \quad \text{for all} \quad \xi, \zeta \in \Lambda' \]

(H2) \[ |\xi| \leq \beta \quad \text{for almost every} \quad x \in \Omega \]

(H3) \[ |A(x, \xi)| \leq \alpha \quad \text{for all} \quad x \in \Omega \text{ and almost every} \quad \xi \in \Lambda' \]

for almost every \( x \in \Omega \), \( \beta \in R \) and all \( \xi, \zeta \in \Lambda' \). Here \( \alpha, \beta > 0 \) are constants and \( |\xi| \) is a fixed exponent associated with (1.1).

When \( g = 0 \) and \( d'f = 0 \), equation (1.1) becomes

\[
d'A(x, du) = 0 \tag{1.3}
\]

There has been remarkable work [1-10] in the study of the equation (1.3). When \( u \) is a 0-form, that is, \( u \) is a function, (1.3) is equivalent to

\[
\text{div}(A(x, V_u)) = 0 \tag{1.4}
\]

Lots of results have been obtained in recent years about different versions of the A-harmnic equation, see [11-15].

In 1995, B. Stroffolini [16] first introduced weakly A-harmonic tensors and given the higher integrability result of weakly A-harmonic tensors. The word weak means that the integrable exponent \( p \) of \( u \) is smaller than the natural exponent \( p \).

Definition 1.1 [16] A very weak solution to (1.1) (also called weakly A-harmonic tensor) is an element of the Sobolev space \( W^{1, r}(\Omega, \Lambda') \) with \( \max \{1, p - 1\} \leq \beta \) such that

\[
\int_{\Omega} (A(x, g + du), d\varphi) dx = \int_{\Omega} (f, d\varphi) dx
\]

for all \( g \in W^{1, r}(\Omega, \Lambda') \) with compact support.

In this paper, we continue to consider the weakly A-harmonic tensor. Based on the weak reverse Holder inequality of weakly A-harmonic tensor, we establish the imbedding inequalities and Poincare inequality of weakly A-harmonic tensors.

II. NOTION AND LEMMAS

We keep using the traditional notation.

Let \( \Omega \) be a connected open subset of \( R^n \). \( e_i, e_2, \ldots, e_n \) be the standard unit basis of \( R^n \), and \( \Lambda^l(\Omega) \) be the linear space of \( l \)-covectors, spanned by the exterior products \( e_i \wedge e_j \wedge \cdots \wedge e_l \) corresponding to all ordered tuples \( \{l \} \leq l \leq n \).

The Grassman algebra \( \Lambda = \oplus \Lambda^l \) is a graded algebra with respect to the exterior products. For \( \alpha = \sum \alpha \xi e \wedge \eta \), \( \beta = \sum \beta \xi e \wedge \eta \), we define the inner product in \( \Lambda \) as given by \( \alpha \wedge \beta = \sum \alpha \beta \xi e \wedge \eta \), and \( \alpha \wedge \beta = \alpha \wedge \eta = \eta \wedge \alpha = 0 \) for all \( \alpha, \beta \in \Lambda \).

The norm of \( \alpha \in \Lambda \) is given by the formula

\[
|\alpha| = \sum |\alpha (x, e) e \wedge \cdots \wedge e_n| R^n \tag{1.2}
\]

The Hodge star is an isometric isomorphism on \( \Lambda \) with \( \ast_a \wedge \Lambda^l \to \Lambda^l \). Balls are denoted by \( B \) and \( \rho \) is the ball with the same center as \( B \) and with diam \( \rho B = \rho \text{diam}(B) \). We do not distinguish balls from cubes throughout this paper. The \( |E| \)-dimensional Lebesgue measure of a set \( E \subset R^n \) is denoted by \( |E| \).

Differential forms are important generalizations of real functions and distributions, note that a \( 0 \)-form is the usual function in \( R^n \). A differential \( l \)-form \( \omega \) on \( \Omega \) is a Schwartz distribution on \( \Omega \) with values in \( \Lambda^l(\Omega) \). We use \( D'(\Omega, \Lambda^l) \) to denote the space of all differential \( l \)-forms \( \omega(x) = \sum \omega(x, e) e \wedge \cdots \wedge e_n \). We write \( L^l(\Omega) \) for the \( l \)-forms with \( \omega \in L^l(\Omega, \Lambda^l) \) for all ordered \( l \)-tuples \( I \). Thus \( L^0(\Omega, \Lambda^0) \) is a Banach space with norm

\[
\|\omega\|_{L^l} = \left( \int_{\Omega} |\omega(x)|^l dx \right)^{1/l} = \left( \int_{\Omega} \left( \sum |\omega(x, e)|^l \right)^{1/l} dx \right)^{1/l}
\]

For \( \omega \in D'(\Omega) \) the vector-valued differential form \( \nabla \omega = (\frac{\partial \omega}{\partial x_1}, \cdots, \frac{\partial \omega}{\partial x_n}) \) consists of differential forms \( \frac{\partial \omega}{\partial x_i} \in D'(\Omega, \Lambda^1) \) where the partial differentials are applied to the coefficients of \( \omega \). As usual, \( W^{1, l}(\Omega) \) is used to denote the Sobolev space of \( l \)-forms, which equals \( L^l(\Omega) \cap L^{1, l}_w(\Omega) \) with norm

\[
\|\omega\|_{W^{1, l}(\Omega, \Lambda^l)} = \text{diam}(\Omega)^{1/l} \|\omega\|_{L^l} + \|\nabla \omega\|_{L^l}.
\]

The notations \( W^{1, l}(\Omega, R) \) and \( W^{1, l}(\Omega, \Lambda^l) \) are self-explanatory. We denote the exterior derivative by
Suppose that \( f \) and \( g \), \( 1 \leq l \leq n \), are measurable functions on \( \Omega \). Let \( u \in \mathcal{L}_{1,1}^{1,1}(\Omega, A) \) be a weakly A-harmonic tensor and \( v \) be a very weak A-harmonic tensor satisfying (1.1) in a bounded domain \( \Omega \). Then, by (3.4) and Lemma 2.2, we have
\[
\int_{\Omega} u \cdot v dx \leq C(n, p)(\int_{\Omega} |u|^p dx)^{1/p} (\int_{\Omega} |v|^{p'} dx)^{1/p'}
\]
for some \( C(n, p) \). Further, we need the following weak reverse Holder inequality.

**Lemma 2.2** Let \( 0 < \alpha < \infty, 0 < \beta < \infty \) and \( s^{-1} = \alpha^{-1} + \beta^{-1} \). If \( f \) and \( g \) are measurable functions on \( R^r \), then
\[
|| f ||_{\alpha, \beta} \leq || f ||_{\alpha, \beta} \leq || g ||_{\alpha, \beta}
\]
for any concentric cubes \( \Omega \subset R^r \).

We need the following weak reverse Holder inequality of weakly A-harmonic tensors.

**Lemma 2.3** Given the A-harmnic equation (1.1), let \( \varepsilon = c(n, p, \alpha, \beta) \in (0, p-1) \). Suppose that \( u \in W^{1,1}(\Omega, \kappa^{-1}) \) is an weakly A-harmonic tensor for some \( \kappa \in (p - e, p) \). Then for any concentric cubes \( \Omega \subset 2Q \subset \Omega \), we have
\[
\left( \int_{\Omega} |u|^p dx \right)^{1/p} \leq C(n, p)(\int_{2Q} |u|^p dx)^{1/p'}
\]
where
\[
r_2 = \max \left[ \frac{nr}{n-r+1}, \frac{nr}{np-n-r+p+1} \right] > 0.
\]
Here \( r_2 < r, 1 < p < \infty \), the constant \( C(n, p) \) does not depend on \( r \) and \( r_2 \).

**III. INEQUALITIES OF WEAKLY A-HARMONIC TENSOR**

Now, we prove the following imbedding inequality of weakly A-harmonic tensor.

**Theorem 3.1** Let \( u \in W_{1,1}^1(\Omega, \kappa^{-1}), 1 \leq l \leq n \), \( \max[1, p-1] \leq r < p \), be a weakly A-harmonic tensor satisfying (1.1) in a bounded domain \( \Omega \subset R^r \) and \( T : C^\infty(\Omega, \kappa^{-1}) \to C^\infty(\Omega, \kappa^{-1}) \) be a homotopy operator. Then there exists a constant \( C \) independent of \( u \) such that
\[
\left( \int_{\Omega} |T(u)|^{p} dx \right)^{1/p} \leq C(n, p)B(\int_{\Omega} |u|^{p} dx)^{1/p'}
\]
for all balls \( B \) with \( 2B \subset \Omega \), where \( s \leq r \),
\[
s = \max \left[ \frac{nr}{n-r+1}, \frac{nr}{np-n-r+p+1} \right] > 0.
\]

**Proof** Let \( u \in W_{1,1}^1(\Omega, \kappa^{-1}), 1 \leq l \leq n \), be a very weak solution of (1.1). By Lemma 2.1 and Lemma 2.3, we have
\[
\left( \int_{\Omega} |\nabla(T(u))|^{p} dx \right)^{1/p} = C(n, p)B(\int_{\Omega} |u|^{p} dx)^{1/p'}
\]
for all \( s < r \) by (3.4) and Lemma 2.2, we have
\[
\left( \int_{\Omega} |\nabla(T(u))|^{p} dx \right)^{1/p} \leq C(n, p)B(\int_{\Omega} |u|^{p} dx)^{1/p'},
\]
where \( s \) is as in (3.3). Note that (3.4) can be written as
\[
\left( \int_{\Omega} |\nabla(T(u))|^{p} dx \right)^{1/p} \leq C(n, p)B(\int_{\Omega} |u|^{p} dx)^{1/p'},
\]
and
\[
\left( \int_{\Omega} |\nabla(T(u))|^{p} dx \right)^{1/p} \leq C(n, p)B(\int_{\Omega} |u|^{p} dx)^{1/p'}
\]
for \( s < r \) by (3.4) and Lemma 2.2, we have
\[
\left( \int_{\Omega} |\nabla(T(u))|^{p} dx \right)^{1/p} \leq C(n, p)B(\int_{\Omega} |u|^{p} dx)^{1/p'}.
\]
Now we prove the following Poincare-type inequality for $T(\omega)$ with the $L'$-norm which plays an important role in this paper.

**Theorem 3.2** Let $u \in W_{loc}^{1,n}(\Omega,\Lambda^{\omega-1})$, \( l = 1, 2, \ldots, n \),

\[
\max \{ \| p - 1 \| \leq r < p \}, \text{ be a weakly A-harmonic tensor satisfying (1.6) in a bounded domain } \Omega \subset \mathbb{R}^n \text{ and } T: C^2(\Omega,\Lambda^\omega) \rightarrow C^0(\Omega,\Lambda^{\omega-1}) \text{ be a homotopy operator. Then, there exists a constant } C \text{ independent of } u \text{ such that }
\]

\[
\int_B |T u - (T u)_B| ^r \, dx = C \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | dx | \right)^{r} \leq C(n, p) \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | dx | \right)^{r},
\]

for all balls $B$ with $2B \subset \Omega$, where $\omega$ is as in (3.3), $s < r$.

**Proof** Let $u \in W_{loc}^{1,n}(\Omega,\Lambda^{\omega-1})$ be a very weak solution of (1.6). By Lemma 2.1 and Lemma 2.3, we have

\[
\int_{\Omega} | T(du) | ^r \, dx = | T(du) | _{L^r(\Omega)},
\]

\[
\leq C \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | dx | \right)^{r} \leq C(n, p) \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | dx | \right)^{r},
\]

where $s$ is as in (3.3). Note that (3.9) can be written as

\[
\int_{\Omega} | T(du) | ^r \, dx \leq C(n, p) \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | dx | \right)^{r}.
\]

For $T u - (T u)_B = T d(T u)_B$, we have

\[
\int_B |T u - (T u)_B| ^r \, dx = C(n, p) \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | dx | \right)^{r}
\]

\[
\leq C(n, p) \cdot | B | \cdot \text{diam}(B) \left( \frac{1}{r} \int_B | d(T u)| \, dx \right)^{r},
\]

for all balls $B$ with $2B \subset \Omega$, where $\omega$ is as in (3.3).

**REFERENCES**


