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Imbedding Inequalities and Poincare Inequalities of Weakly A-harmonic Tensors

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***Abstract* — The Imbedding Inequalities and Poincare Inequalities of Weakly A-harmonic tensors have been proved.**

***Keywords* — Weakly A-Harmonic Tensor, Differential Form, Imbedding Inequality, Poincare Inequalities.**

I. Introduction

Given $g \in L^r(\Omega,\wedge^l)$ and $f \in L^{r/(p-1)}(\Omega,\wedge^l)$ where $r \ge \max\{1, p-1\}$, we consider the non homogeneous equation for differential forms

$$d^*A(x,g+du) = d^*f \text{ for } u \in W^{1,r}_{loc}(\Omega,\wedge^l) \quad (1.1)$$

Where $A:\Omega\times\wedge^l(R^n)\to\wedge^{l+1}(R^n)$ satisfies the conditions

(H1) $|A(x,\xi)-A(x,\zeta)| \le \beta|\xi-\zeta|(|\xi|+|\zeta|)^{p-2}$,

(H2) $< A(x,\xi)-A(x,\zeta), \xi-\zeta > \ge \alpha|\xi-\zeta|^2(|\xi|+|\zeta|)^{p-2}$,

(H3) $A(x,\lambda\xi) = |\lambda|^{p-2}\lambda A(x,\zeta)$, (1.2)

for almost every $x\in\Omega$, $\lambda\in R$ and all $\xi,\zeta\in\wedge^l(R^n)$. Here $\alpha,\beta>0$ are constants and $1<p<\infty$ is a fixed exponent associated with (1.1).

When $g=0$ and $d^*f=0$, equation (1.1) becomes

$$d^*A(x,du)=0. \quad (1.3)$$

There has been remarkable work [1–10] in the study of the equation (1.3). When u is a 0-form, that is, u is a function, (1.3) is equivalent to

$$\operatorname{div}A(x,\nabla u)=0. \quad (1.4)$$

Lots of results have been obtained in recent years about different versions of the A-harmnic equation, see [11-15].

In 1995, B. Stroffolini [16] first introduced weakly A-harmonic tensors and given the higher integrability result of weakly A-harmonic tensors. The word *weak* means that the integrable exponent r of u is smaller than the natural exponent p.

Definition 1.1[16] A very weak solution to (1.1) (also called weakly A-harmonic tensor) is an element u of the Sobolev space $W^{1,r}_{loc}(\Omega,\wedge^{l-1})$ with $\max\{1,p-1\}\le r<p$ such that

$$\int_\Omega \langle A(x,g+du),d\varphi\rangle dx = \int_\Omega \langle f,d\varphi\rangle dx \quad (1.5)$$

for all $\varphi\in W^{1,\frac{r}{r-p+1}}(\Omega,\wedge^{l-1})$ with compact support.

In this paper, we continue to consider the weakly A-harmonic tensor. Based on the weak reverse Holder inequality of weakly A-harmonic tensor, we establish the imbedding inequalities and Poincare inequality of weakly A-harmonic tensors.

II. Notion And Lemmas

We keep using the traditional notation.

Let Ω be a connected open subset of R^n, $e_1,e_2,\cdots,e_n$ be the standard unit basis of R^n,and $\wedge^l=\wedge^l(R^n)$ be the linear space of $l-$covectors , spanned by the exterior products $e_I=e_{i1}\wedge e_{i2}\wedge\cdots\wedge e_{il}$,corresponding to all ordered $l-$tuples $I=(i_1,i_2,\cdots,i_l)$, $1\le i_1<i_2<\cdots<i_l\le n$, $l=0,1,\cdots,n$. The Grassman algebra $\wedge=\oplus\wedge^l$ e is a graded algebra with respect to the exterior products. For $\alpha=\sum\alpha^I e_I\in\wedge$ and $\beta=\sum\beta^I e_I\in\wedge$, the inner product in $\wedge$ is given by $<\alpha,\beta>=\sum\alpha^I\beta^I$ with summation over all $l-$ tuples $I=(i_1,i_2,\cdots,i_l)$ and all integers $l=0,1,\cdots,n$. We define the Hodge star operator $*:\wedge\to\wedge$ by the rule $*1=e_1\wedge e_2\wedge\cdots\wedge e_n$, and $\alpha\wedge*\beta=\beta\wedge*\alpha=<\alpha,\beta>(*1)$ for all α, $\beta\in\wedge$. The norm of $\alpha\in\wedge$ is given by the formula $|\alpha|^2=<\alpha,\alpha>=*(\alpha\wedge*\alpha)\in\wedge^0=R$. The Hodge star is an isometric isomorphism on $\wedge$ with $*:\wedge^l\to\wedge^{n-l}$ and $**(-1)^{l(n-l)}:\wedge^l\to\wedge^l$. Balls are denoted by B and ρB is the ball with the same center as B and with $\operatorname{diam}(\rho B)=\rho\operatorname{diam}(B)$. We do not distinguish balls from cubes throughout this paper. The $n-$ dimensional Lebesgue measure of a set $E\subseteq R^n$ is denoted by $|E|$.

Differential forms are important generalizations of real functions and distributions, note that a $0-$ form is the usual function in R^n. A differential $l-$form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(R^n)$. We use $D'(\Omega,\wedge^l)$ to denote the space of all differential $l-$forms $\omega(x)=\sum_I\omega_I(x)dx_I=\sum\omega_{i_1i_2\cdots i_l}(x)dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_l}$. We write $L^p(\Omega,\wedge^l)$ for the $l-$forms with $\omega_I\in L^p(\Omega,R)$ for all ordered $l-$tuples I. Thus $L^p(\Omega,\wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega}=(\int_\Omega|\omega(x)|^p dx)^{1/p}=(\int_\Omega(\sum|\omega_I(x)|^2)^{p/2}dx)^{1/p}.$$

For $\omega\in D'(\Omega,\wedge^l)$ the vector-valued differential form $\nabla\omega=(\frac{\partial\omega}{\partial x_1},\cdots,\frac{\partial\omega}{\partial x_n})$ consists of differential forms $\frac{\partial\omega}{\partial x_i}\in D'(\Omega,\wedge^l)$ where the partial differentials are applied to the coefficients of ω. As usual, $W^{1,p}(\Omega,\wedge^l)$ is used to denote the Sobolev space of $l-$ forms, which equals $L^p(\Omega,\wedge^l)\cap L_1^p(\Omega,\wedge^l)$ with norm

$$\|\omega\|_{W^{1,p}(\Omega,\wedge^l)}=\operatorname{diam}(\Omega)^{-1}\|\omega\|_{p,\Omega}+\|\nabla\omega\|_{p,\Omega}.$$

The notations $W^{1,p}_{loc}(\Omega,R)$ and $W^{1,p}_{loc}(\Omega,\wedge^l)$ are self - explanatory. We denote the exterior derivative by

$d: D'(\Omega,\wedge^l)\to D'(\Omega,\wedge^{l+1})$ for $l=0,1,\cdots,n$. Its formal adjoint operator $d^*: D'(\Omega,\wedge^{l+1})\to D'(\Omega,\wedge^l)$ is given by $d^*=(-1)^{nl+1}*d*$ on $D'(\Omega,\wedge^{l+1})$, $l=0,1,\cdots,n$.

Let $u\in L^1_{loc}(\Omega,\wedge^l)$, $l=0,1,\cdots,n$. We write $u\in locLip_k(\Omega,\wedge^l)$,

$0\le k\le 1$, if

$$\| u\|_{locLip_k,\Omega}=\sup_{\sigma Q\subset\Omega}|Q|^{-(n+k)/n}\| u-u_Q\|_{1,Q}<\infty \tag{2.1}$$

for some $\sigma\ge 1$. Further, we write $Lip_k(\Omega,\wedge^l)$, for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\| u\|_{Lip_k,\Omega}$ for this norm. Similarly, for $u\in L^1_{loc}(\Omega,\wedge^l)$, $l=0,1,\cdots,n$, we write $u\in BMO(\Omega,\wedge^l)$ if

$$\| u\|_{*,\Omega}=\sup_{\sigma Q\subset\Omega}|Q|^{-1}\| u-u_Q\|_{1,Q}<\infty \tag{2.2}$$

for some $\sigma\ge 1$. When u is a $0-$form, (1.2) reduces to the classical definition of $BMO(\Omega)$.

From [1, 18], if $D\subset R^n$ be a bounded, convex domain, to each $y\in D$ there corresponds a linear operator $K_y: C^\infty(D,\wedge^l)\to C^\infty(D,\wedge^{l-1})$ defined by

$$(K_y\omega)(x;\xi_1,\cdots,\xi_{l-1})=\int_0^1 t^{l-1}\omega(tx+y-ty;x-y,\xi_1,\cdots,\xi_{l-1})dt$$

and a decomposition $\omega=d(K_y\omega)+K_y(d\omega)$.

A homotopy operator $T: C^\infty(D,\wedge^l)\to C^\infty(D,\wedge^{l-1})$ is defined by averaging K_y over all points y in D, i.e.,

$$T\omega=\int_D\varphi(y)K_y\omega dy, \tag{2.3}$$

where $\varphi\in C_0^\infty(D)$ is normalized by $\int_D\varphi(y)dy=1$. Then, there is also a decomposition

$$\omega=d(T\omega)+T(d\omega). \tag{2.4}$$

The $l-$form $\omega_D\in D'(D,\wedge^l)$ is defined by

$$\omega_D=\begin{cases}|D|^{-1}\int_D\omega(y)dy & if\ l=0\\ d(T\omega) & if\ l=1,2,\cdots,n\end{cases}$$

for all $\omega\in L^p(D,\wedge^l)$, $1\le p<\infty$. Then $\omega_D=\omega-T(d\omega)$. Clearly ω_D is a closed form and for $l>0$, ω_D is an exact form. By substituting $z=tx+y-ty$, (1.3) reduces to

$$T\omega(x,\xi)=\int_D\omega(z,\zeta(z,x-z,\xi))dz, \tag{2.5}$$

Where the vector function $\zeta: D\times R^n\to R^n$ is given by

$$\zeta(z,h)=h\int_0^\infty s^{l-1}(1+s)^{n-1}\varphi(z-sh)ds.$$

Integral (2.5) defines a bounded operator $T: L^s(D,\wedge^l)\to W^{1,s}(D,\wedge^{l-1})$, $l=1,2,\cdots,n$. with norm estimated by

$$\| Tu\|_{W^{1,s}(D)}\le C|D|\| u\|_{s,D}.$$

From results appearing in [18], we have the following lemma.

Lemma 2.1 Let $u\in L^s_{loc}(B,\wedge^l)$, $l=1,2,\cdots,n$, $1<s<\infty$, be a differential form in a ball $B\subset R^n$.

$$\|\nabla(Tu)\|_{s,B}\le C|B|\| u\|_{s,B},$$
$$\| Tu\|_{s,B}\le C|B|\mathrm{diam}(B)\| u\|_{s,B}.$$

We will need the following generalized Holder inequality.

Lemma 2.2 Let $0<\alpha<\infty, 0<\beta<\infty$ and $s^{-1}=\alpha^{-1}+\beta^{-1}$. If f and g are measurable functions on R^n, then

$$\| fg\|_{s,\Omega}\le\| f\|_{\alpha,\Omega}\cdot\| g\|_{\beta,\Omega}$$

For any $\Omega\subset R^n$.

We need the following weak reverse Holder inequality of weakly A-harmonic tensors.

Lemma 2.3 Given the A-harmnic equation (1.1), let $\varepsilon=\varepsilon(n,p,\alpha,\beta)\in(0,p-1)$. Suppose that $u\in W^{1,r_1}(\Omega,\wedge^{l-1})$ is an weakly A-harmonic tensor for some $r_1\in(p-\varepsilon,p)$. Then for any concentric cubes $Q\subset 2Q\subset\Omega$, we have

$$\left(\int_Q|du|^{r_1}\right)^{1/r_1}\le C(n,p)\left(\int_{2Q}|du|^{r_2}\right)^{1/r_2}$$

Where

$$r_2=\max\left\{\frac{nr_1}{n+r_1-1},\frac{nr_1}{np-n+r_1-p+1}\right\}.$$

Here $r_2<r_1$, $1<p<\infty$, the constant $C(n,p)$ does not depend on r_1 and r_2.

III. Inequalities of Weakly A-Harmonic Tensor

Now, we prove the following imbedding inequality of weakly A-harmonic tensor.

Theorem 3.1 Let $u\in W^{1,r_1}_{loc}(\Omega,\wedge^{l-1})$, $l=1,2,\cdots,n$, $\max\{1,p-1\}\le r<p$, be a weakly A-harmonic tensor satisfying (1.1) in a bounded domain $\Omega\subset R^n$ and $T: C^\infty(\Omega,\wedge^l)\to C^\infty(\Omega,\wedge^{l-1})$ be a homotopy operator. Then there exists a constant C independent of u such that

$$\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}\le C(n,p)|B|\left(\int_{2B}|du|^r\right)^{1/r}, \tag{3.1}$$

$$\left(\int_B|T(du)|^r dx\right)^{1/r}\le C(n,p)|B|\left(\int_{2B}|du|^s\right)^{1/s}, \tag{3.2}$$

for all balls B with $2B\subset\Omega$, where $s<r$,

$$s=\max\left\{\frac{nr}{n+r-1},\frac{nr}{np-n+r-p+1}\right\}. \tag{3.3}$$

Proof Let $u\in W^{1,r_1}_{loc}(\Omega,\wedge^{l-1})$, $l=1,2,\cdots,n$, be a very weak solution of (1.1). By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}&=\|\nabla(T(du))\|_{r,B}\\ &\le C|B|\| du\|_{r,B}\\ &=C|B|\left(\int_B|du|^r dx\right)^{1/r}\\ &\le C(n,p)|B\|B|^{1/r}\left(\int_{2B}|du|^s\right)^{1/s},\end{aligned} \tag{3.4}$$

where s is as in (3.3). Note that (3.4) can be written as

$$\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}\le C(n,p)|B|\left(\int_{2B}|du|^r\right)^{1/r}. \tag{3.5}$$

For $s<r$, by (3.4) and Lemma 2.2, we have

$$\begin{aligned}&\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}\\ &\le C(n,p)|B\|B|^{1/r}|B|^{-1/s}\left(\int_{2B}|du|^s\right)^{1/s}\\ &\le C(n,p)|B\|B|^{1/r}|B|^{-1/s}|B|^{\frac{r-s}{rs}}\left(\int_{2B}|du|^r\right)^{1/r}\\ &=C(n,p)|B|\left(\int_{2B}|du|^r\right)^{1/r}.\end{aligned} \tag{3.6}$$

Now we prove the following Poincare-type inequality for $T(u)$ with the L^s – norm which plays an important role in this paper.

Theorem 3.2 Let $u \in W^{1,r_1}_{loc}(\Omega, \wedge^{l-1})$, $l = 1,2,\cdots,n$, $\max\{1, p-1\} \le r < p$, be a weakly A-harmonic tensor satisfying (1.6) in a bounded domain $\Omega \subset R^n$ and $T: C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator. Then, there exists a constant C independent of u such that

$$(\int_B |Tu-(Tu)_B|^r\, dx)^{1/r} \le C(n,p)\,|B|\,\mathrm{diam}(B)(\int_{2B} |u|^s dx)^{1/s}, \quad (3.7)$$

$$(\int_B |Tu-(Tu)_B|^r\, dx)^{1/r} \le C(n,p)\,|B|\,\mathrm{diam}(B)(\int_{2B} |u|^r dx)^{1/r}, \quad (3.8)$$

for all balls B with $2B \subset \Omega$, where s is as in (3.3), $s < r$.

Proof Let $u \in W^{1,r_1}_{loc}(\Omega, \wedge^{l-1})$ be a very weak solution of (1.6). By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}(\int_B |T(du)|^r\, dx)^{1/r} &= \|T(du)\|_{r,B} \\ &\le C\,|B|\,\mathrm{diam}(B)\,\|du\|_{r,B} \\ &= C\,|B|\,\mathrm{diam}(B)(\int_B |du|^r\, dx)^{1/r} \\ &\le C(n,p)\,|B|\,diam(B)\,|B|^{1/r}\,(\int_{2B} |du|^s)^{1/s},\end{aligned} \quad (3.9)$$

where s is as in (3.3). Note that (3.9) can be written as

$$(\int_B |T(du)|^r\, dx)^{1/r} \le C(n,p)\,|B|\,\mathrm{diam}(B)(\int_{2B} |du|^s)^{1/s}. \quad (3.10)$$

For $Tu-(Tu)_B = Td(Tu)$, we have

$$\begin{aligned}&(\int_B |Tu-(Tu)_B|^r\, dx)^{1/r} \\ &= (\int_B |Td(Tu)|^r\, dx)^{1/r} \\ &\le C(n,p)\,|B|\,\mathrm{diam}(B)(\int_{2B} |d(Tu)|^s dx)^{1/s}\end{aligned} \quad (3.11)$$

for all balls B with $2B \subset \Omega$, where s is as in (3.3).

REFERENCES

[1] R. P. Agarwal, S. Ding, and C. Nolder. Inequalities for Differential Forms. Springer, New York, NY, USA, 2009.

[2] C. A. Nolder. Hardy-Littlewood theorems for A-harmonic tensors. Illinois Journal of Mathematics, vol. 43, no. 4, pp. 613-632, 1999.

[3] T. Iwaniec, G. Martin. Quasi regular mappings in even dimensions. Acta Math., 170 (1993), 29-81

[4] S. Ding, Jianmin Zhu. Poincare-type inequalities for the homotopy operator with Lϕ(m)-norms. Nonlinear Analysis: Theory, Methods and Applications, Volume 74, Issue 11, July 2011, Pages 3728 C 3735

[5] S. Ding, C.A. Nolder. Weighted Poincare inequalities for solutions to A-harmonic equations. Illinois Journal of Mathematics, Volume 46, Number 1 (2002), 199-205.

[6] Wang Yong, Li Guanfeng. Weighted Decomposition Estimates for Differential Forms. Journal of Inequalities and Applications, Volume 2010, Article ID 649340, 17 pages, doi:10.1155/2010/649340

[7] Yuming Xing. Poincare Inequalities with Luxemburg Norms in Lϕ(m)-Averaging Domains. Journal of Inequalities and Applications, vol. 2010, Article ID 241759, 11 pages, 2010. doi:10.1155/2010/241759

[8] Hui Bi. Weighted Inequalities for Potential Operators on Differential Forms. Journal of Inequalities and Applications, vol. 2010, Article ID 713625, 13 pages, 2010. doi:10.1155/2010/713625.

[9] Hongya Gao. Weighted integral inequalities for conjugate A-harmonic tensors. Journal of Mathematical Analysis and Applications, Volume 281, Issue 1, 1 May 2003, Pages 253-263.

[10] Yuxia Tong, Juan Li, Jiantao Gu. Aλ3 r (λ1,λ2,Ω)-Weighted Inequalities with Lipschitz and BMO Norms. Journal of Inequalities and Applications, Volume 2010, Article ID 713625, 13 pages, doi:10.1155/2010/713625.

[11] J. Heinonen, T. Kilpelainen and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press Inc., New York, 1993.

[12] Gongbao Li, Caiyun Yang. The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition, Nonlinear Analysis, 72 (2010), 4602-4613

[13] Shenzhou Zheng, Xueliang Zheng, Zhaosheng Feng. Optimal regularity for A-harmonic type equations under the natural growth. Discrete and Continuous Dynamical Systems(Series B), pp.669 -685, Volume 16, Issue 2, September 2011

[14] Hongya Gao, Jinjing Qiao, Yuming Chu. Local Regularity and Local Boundedness results for Very Weak Solutions of Obstacle Problems. Journal of Inequalities and Applications, vol. 2010, Article ID 878769, 2010. doi:10.1155/2010/878769

[15] Gejun Bao, Tingting Wang, Guanfeng Li. On very weak solutions to a class of double obstacle problems. J. Math. Anal. Appl., 402 (2013) 702-709

[16] B. Stroffolini. On weakly A-harmonic tensors, Studia Math, 1995, 114(3): 289-301.

[17] Hongya Gao, Yanyan Wang. Weak WT2-class of differential forms and weakly A-harmonic tensors, Appl. Math. J. Chinese Univ. 2010, 25(3): 359-366.

[18] T. Iwaniec and A. Lutoborski, Integral estimates for null Lagrangians, Arch. Rational Mech. Anal., 125 (1993), 25-79.

[19] T. Iwaniec, L. Migliaccio, L. Nania, C. Sbordone. Integrability and removability results for quasiregular mappings in high dimensions, Math Scand, 1994, 75: 263-279.

[20] M Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Ann of Math Stud, 105, Princeton University Press, 1983.

[21] T. Iwaniec, G. Martin. Geometric function theory and nonlinear analysis [M]. Oxford: Clarendon Press 2001.G. O. Young, “Synthetic structure of industrial plastics (Book style with paper title and editor),” in Plastics, 2nd ed. vol. 3, J. Peters, Ed. New York: McGraw-Hill, 1964, pp. 15–64. (8)