

Advancing Hemiring Theory through S-Fuzzysoft Subhemiring Structures

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Abstract – In this manuscript, we introduce and investigate the concept of *lower-level subhemirings* within the context of *S-fuzzy soft subhemirings* of a hemiring, grounded in the theoretical framework of soft set theory. This study extends existing algebraic structures by incorporating fuzziness and parameterization, which provide greater flexibility in handling uncertainty and imprecision. The notion of lower-level subhemirings is developed to identify and characterize specific substructures that emerge from the interaction between fuzzy soft elements and the underlying hemiring. 2020 Mathematics subject classification: 03F55, 03E72, 06D72.

Keywords – \mathcal{S} -Fuzzy, Soft Sub Hemiring, Lower Level Subhemirings, \mathcal{S} -Fuzzy Soft Sub Hemiring, Normal \mathcal{S} -Fuzzy Soft Subhemiring.

I. INTRODUCTION

In our mundane life vulnerability demonstrated up in similar in approach speculate on the grounds that our surroundings are encumbered with vulnerabilities. As a result, it hushes up distinctive us to exhibit this vulnerability winning in physical world. An massive number of these representation depend on an intensification of normal set hypotheses, for example, fuzzy sets, Intuitionistic fuzzy sets, delicate set etc. Dealing with vulnerabilities is a significant issue in numerous zones, for example, financial matters, designing, natural science, medicinal science and sociologies. To defeat these sorts of challenges, Molodtsov [11] proposed a totally new approach, which is called soft set hypothesis, for displaying vulnerability. At that point Maji et al. [9, 10] presented a few tasks on soft sets. Akta and Cagman [1] characterized soft gatherings and acquired the principle properties of these gatherings. Besides, they contrasted delicate sets and fuzzy sets and unpleasant sets. Also, Jun et al. [7] characterized delicate goals on BCK/BCI-algebras. Feng et al. [7, 8] characterized soft semirings, delicate standards on soft semirings and hopeful soft semirings. Qiu-Mei Sun et al. characterized the idea of soft modules and concentrated their essential properties. The principle reason for this paper is to present fundamental ideas of S-fuzzy soft subhemirings of a hemiring, which are really a parameterized group of subhemiring of a over a hemiring R. Besides, the idea of the Lower level S-soft subhemiring homomorphism is presented.

II. PRELIMINARIES

2.1. Definition:

A \mathcal{S} -norm is a binary operation $\mathcal{S}: [0,1] \times [0,1] \rightarrow [0,1]$ satisfying the following requirements:

- (i) $0 \mathcal{S} x = x$ (Boundary conditions).
- (ii) $x \mathcal{S} y = y \mathcal{S} x$ (Commutativity).
- (iii) $x \mathcal{S} (y \mathcal{S} z) = (x \mathcal{S} y) \mathcal{S} z$ (Associativity).

(iv) If $x \leq y$ and $w \leq z$, then $x \leq y \leq z$ (monotonicity).

2.2. Definition:

Let $(\dot{R}, +, \cdot)$ be a hemiring. A fuzzy subset of \dot{R} is said to be a \mathcal{S} -fuzzy subhemiring (fuzzy subhemiring with respect to \mathcal{S} -norm) of \dot{R} if it satisfies the following conditions:

- (i) $\mathcal{F}_{(D,W)}(x + y) \geq \mathcal{S}(\mathcal{F}_{(D,W)}(x), \mathcal{F}_{(D,W)}(y))$
- (ii) $\mathcal{F}_{(D,W)}(xy) \geq \mathcal{S}(\mathcal{F}_{(D,W)}(x), \mathcal{F}_{(D,W)}(y))$, for all x and y in \dot{R} .

2.3. Definition:

Let (D, W) be a \mathcal{S} -fuzzy soft subset of \dot{X} . For α in $[0, 1]$, the lower level soft subset of (D, W) is the set $(D, W)_\alpha = \{x_{(D,W)} \in \dot{X} : \mathcal{F}_{(D,W)}(x_{(D,W)}) \geq \alpha\}$.

2.4. Definition:

Let (D, W) be a \mathcal{S} -fuzzy soft subhemiring of a hemiring \dot{R} . Then $(D, W)^0$ is defined as $(D, W)^0(x_{(D,W)}) = \mathcal{F}_{(D,W)}(x_{(D,W)}) / (D, W)(0)$, for all $x_{(D,W)} \in \dot{R}$.

2.5. Definition:

Let $(\dot{R}, +, \cdot)$ be a hemiring. A \mathcal{S} -fuzzy soft subhemiring (D, W) of \dot{R} is said to be a \mathcal{S} -fuzzy soft normal subhemiring (TFSNSHR) of \dot{R} if $\mathcal{F}_{(D,W)}(x_{(D,W)} y_{(D,W)}) = \mathcal{F}_{(D,W)}(y_{(D,W)} x_{(D,W)})$, for all $x_{(D,W)}$ and $y_{(D,W)}$ in \dot{R} .

III. LOWER LEVEL SUB HEMIRINGS OF A \mathcal{S} -FUZZY SOFT SUBHEMIRING OF A HEMIRING

3.1. Theorem:

Let (D, W) be a \mathcal{S} -fuzzy soft subhemiring of a hemiring \dot{R} . Then for α in $[0, 1]$, $(D, W)_\alpha$ is a lower level soft subhemiring of \dot{R} .

Proof:

For all $x_{(D,W)}$ and $y_{(D,W)}$ in $(D, W)_\alpha$, we have, $\mathcal{F}_{(D,W)}(x_{(D,W)}) \geq \alpha$ and $\mathcal{F}_{(D,W)}(y_{(D,W)}) \geq \alpha$.

Now, $\mathcal{F}_{(D,W)}(x_{(D,W)} + y_{(D,W)}) \geq \mathcal{S}\{\mathcal{F}_{(D,W)}(x_{(D,W)}), \mathcal{F}_{(D,W)}(y_{(D,W)})\} \geq \mathcal{S}\{\alpha, \alpha\} = \alpha$, which implies that $\mathcal{F}_{(D,W)}(x_{(D,W)} + y_{(D,W)}) \geq \alpha$.

And, $\mathcal{F}_{(D,W)}(x_{(D,W)} y_{(D,W)}) \geq \mathcal{S}\{\mathcal{F}_{(D,W)}(x_{(D,W)}), \mathcal{F}_{(D,W)}(y_{(D,W)})\} \geq \mathcal{S}\{\alpha, \alpha\} = \alpha$, which implies that $\mathcal{F}_{(D,W)}(x_{(D,W)} y_{(D,W)}) \geq \alpha$.

Therefore, $\mathcal{F}_{(D,W)}(x_{(D,W)} + y_{(D,W)}) \geq \alpha$ and $\mathcal{F}_{(D,W)}(x_{(D,W)} y_{(D,W)}) \geq \alpha$.

Therefore, $x_{(D,W)} + y_{(D,W)}$ and $x_{(D,W)} y_{(D,W)}$ in $(D, W)_\alpha$.

Hence $(D, W)_\alpha$ is a lower level subhemiring of a hemiring \dot{R} .

3.2. Theorem:

Let (D, W) be a \mathcal{S} -fuzzy soft subhemiring of a hemiring \dot{R} . Then two lower level soft subhemiring $(D, W)_{\alpha_1}$,

$(D, W)_{\alpha_2}$ and α_1, α_2 are in $[0, 1]$ with $\alpha_1 < \alpha_2$ of (D, W) are equal if and only if there is no x in \dot{R} such that $\alpha_2 > \mathfrak{F}_{(D, W)}(x_{(D, W)}) > \alpha_1$.

Proof:

Assume that $(D, W)_{\alpha_1} = (D, W)_{\alpha_2}$.

Suppose there exists $x_{(D, W)}$ in \dot{R} such that $\alpha_2 > \mathfrak{F}_{(D, W)}(x_{(D, W)}) > \alpha_1$.

Then $(D, W)_{\alpha_1} \subseteq (D, W)_{\alpha_2}$ implies x belongs to $(D, W)_{\alpha_2}$, but not in $(D, W)_{\alpha_1}$.

This is contradiction to $(D, W)_{\alpha_1} = (D, W)_{\alpha_2}$.

Therefore there is no $x_{(D, W)} \in \dot{R}$ such that $\alpha_2 > \mathfrak{F}_{(D, W)}(x_{(D, W)}) > \alpha_1$. On the other hand if there is no $x_{(D, W)} \in \dot{R}$ such that $\alpha_2 > \mathfrak{F}_{(D, W)}(x_{(D, W)}) > \alpha_1$.

Then $(D, W)_{\alpha_1} = (D, W)_{\alpha_2}$. by the definition of lower level soft set.

3.3. Theorem:

Let (D, W) be a \S - fuzzy soft subhemiring of a hemiring \dot{R} . If any two lower level soft subhemirings of (D, W) belongs to \dot{R} , then their intersection is also lower level soft subhemiring of (F, A) in \dot{R} .

Proof:

Let $\alpha_1, \alpha_2 \in [0, 1]$.

Case (i):

If $\alpha_1 < \mathfrak{F}_{(D, W)}(x_{(D, W)}) < \alpha_2$, then $(D, W)_{\alpha_1} \subseteq (D, W)_{\alpha_2}$.

Therefore, $(D, W)_{\alpha_1} \cap (D, W)_{\alpha_2} = (D, W)_{\alpha_1}$, but $(D, W)_{\alpha_1}$ is a lower level soft subhemiring of (D, W) .

Case (ii):

If $\alpha_1 > \mathfrak{F}_{(D, W)}(x_{(D, W)}) > \alpha_2$, then $(D, W)_{\alpha_2} \subseteq (D, W)_{\alpha_1}$.

Therefore, $(D, W)_{\alpha_1} \cap (D, W)_{\alpha_2} = (D, W)_{\alpha_2}$, but $(D, W)_{\alpha_2}$ is a lower level soft subhemiring of (D, W) .

Case (iii):

If $\alpha_1 = \alpha_2$, then $(D, W)_{\alpha_1} = (D, W)_{\alpha_2}$.

In all cases, intersection of any two lower level soft subhemirings is a lower level soft subhemiring of (D, W) .

3.4. Theorem:

The homomorphic image of a lower level soft subhemiring of a \S - fuzzy soft subhemiring of a hemiring \dot{R} is a lower level soft subhemiring of a \S - fuzzy soft subhemiring of a hemiring $\dot{R}^!$.

Proof:

Let $(\dot{R}, +, \cdot)$ and $(\dot{R}^!, +, \cdot)$ be any two hemirings and $f: \dot{R} \rightarrow \dot{R}^!$ be a homomorphism.

That is, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in \dot{R} .

Let $(G, V) = f((D, W))$, where (D, W) is an \mathcal{S} -fuzzy soft subhemiring of a hemiring \dot{R} .

Clearly (G, V) is an \mathcal{S} -fuzzy soft subhemiring of a hemiring $\dot{R}^!$.

Let $x_{(D, W)}$ and $y_{(D, W)}$ in \dot{R} , implies $f(x_{(G, V)})$ and $f(y_{(G, V)})$ in $\dot{R}^!$.

Let $(D, W)_\alpha$ is a lower level subhemiring of (D, W) .

That is, $\mathfrak{H}_{(D, W)}(x_{(D, W)}) \geq \alpha$ and $\mathfrak{H}_{(D, W)}(y_{(D, W)}) \geq \alpha$; $\mathfrak{H}_{(D, W)}(x_{(D, W)} + y_{(D, W)}) \geq \alpha$, $\mathfrak{H}_{(D, W)}(x_{(D, W)} y_{(D, W)}) \geq \alpha$.

We have to prove that $f((D, W)_\alpha)$ is a lower level soft subhemiring of (G, V) .

Now, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)})) \geq \mathfrak{H}_{(D, W)}(x_{(D, W)}) \geq \alpha$, which implies that $\mathfrak{H}_{(G, V)}(f(x_{(G, V)})) \geq \alpha$; and $\mathfrak{H}_{(G, V)}(f(y_{(G, V)})) \geq \mathfrak{H}_{(D, W)}(y_{(D, W)}) \geq \alpha$, which implies that $\mathfrak{H}_{(G, V)}(f(y_{(G, V)})) \geq \alpha$ and $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) + f(y_{(G, V)})) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)} + y_{(G, V)}))$, as f is a homomorphism $\geq \mathfrak{H}_{(D, W)}(x_{(D, W)} + y_{(D, W)}) \geq \alpha$, which implies that $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) + f(y_{(G, V)})) \geq \alpha$.

Also, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) f(y_{(G, V)})) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)} y_{(G, V)}))$, as f is a homomorphism $\geq \mathfrak{H}_{(D, W)}(x_{(D, W)} y_{(D, W)}) \geq \alpha$, which implies that $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) f(y_{(G, V)})) \geq \alpha$.

Therefore, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) + f(y_{(G, V)})) \geq \alpha$, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) f(y_{(G, V)})) \geq \alpha$.

Hence $f((D, W)_\alpha)$ is a lower level soft subhemiring of a \mathcal{S} -fuzzy soft subhemiring (G, V) of a hemiring $\dot{R}^!$.

3.5. Theorem:

The homomorphic pre-image of a lower level soft subhemiring of a \mathcal{S} -fuzzy soft subhemiring of a hemiring $\dot{R}^!$ is a lower level soft subhemiring of a \mathcal{S} -fuzzy soft subhemiring of a hemiring \dot{R} .

Proof:

Let $(G, V) = f((D, W))$, where (G, V) is a \mathcal{S} -fuzzy soft subhemiring of a hemiring $\dot{R}^!$.

Clearly (D, W) is a \mathcal{S} -fuzzy soft subhemiring of a hemiring \dot{R} .

Let $f(x_{(G, V)})$ and $f(y_{(G, V)})$ in $\dot{R}^!$, implies $x_{(D, W)}$ and $y_{(D, W)}$ in \dot{R} .

Let $f((D, W)_\alpha)$ is a lower level soft subhemiring of (G, V) .

That is, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)})) \geq \alpha$ and $\mathfrak{H}_{(G, V)}(f(y_{(G, V)})) \geq \alpha$; $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) + f(y_{(G, V)})) \geq \alpha$, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) f(y_{(G, V)})) \geq \alpha$.

We have to prove that $(D, W)_\alpha$ is a lower level soft subhemiring of (D, W) .

Now, $\mathfrak{H}_{(D, W)}(x_{(D, W)}) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)})) \geq \alpha$, implies that $\mathfrak{H}_{(D, W)}(x_{(D, W)}) \geq \alpha$; $\mathfrak{H}_{(D, W)}(y_{(D, W)}) = \mathfrak{H}_{(G, V)}(f(y_{(G, V)})) \geq \alpha$, implies that $\mathfrak{H}_{(D, W)}(y_{(D, W)}) \geq \alpha$ and $\mathfrak{H}_{(D, W)}(x_{(D, W)} + y_{(D, W)}) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)} + y_{(G, V)})) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)}) + f(y_{(G, V)}))$, as f is a homomorphism $\geq \alpha$, which implies that $\mathfrak{H}_{(D, W)}(x_{(D, W)} + y_{(D, W)}) \geq \alpha$.

Also, $\mathfrak{H}_{(D, W)}(x_{(D, W)} y_{(D, W)}) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)} y_{(G, V)})) = \mathfrak{H}_{(G, V)}(f(x_{(G, V)}) f(y_{(G, V)}))$, as f is a homomorphism $\geq \alpha$. Which implies that $\mathfrak{H}_{(D, W)}(x_{(D, W)} y_{(D, W)}) \geq \alpha$.

Therefore, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) + f(y_{(G, V)})) \geq \alpha$, $\mathfrak{H}_{(G, V)}(f(x_{(G, V)}) f(y_{(G, V)})) \geq \alpha$.

Hence, $(D, W)_\alpha$ is a lower level soft subhemiring of a \mathcal{S} -fuzzy soft subhemiring (D, W) of \dot{R} .

III. CONCLUSION

In this work, we have proposed and examined the concept of lower-level subhemirings within the structure of an S-fuzzy soft subhemiring of a hemiring, utilizing the framework of soft set theory. By integrating the soft and fuzzy paradigms, we have extended the algebraic understanding of hemirings and introduced new structural elements that enrich the theoretical landscape of fuzzy soft algebra. Through definitions, examples, and theorems, we have highlighted the key properties and interactions of these lower-level subhemirings, demonstrating their consistency and potential relevance in broader mathematical and applied contexts. The investigation reveals that these structures provide a finer analytical tool for handling uncertainty in algebraic systems, with possible implications for information processing, decision-making, and computational intelligence. This study lays the groundwork for further exploration into advanced soft algebraic systems and encourages the development of related structures in abstract algebra and applied mathematical frameworks.

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