International Journal of Innovation in Science and Mathematics
Volume 2, Issue 3, ISSN (Online): 2347–9051

Generalized Quaternions: Further Contributions to a Matrix Approach and Frenet-Serret Formulas

Faik BABADAĞ

Abstract **– In this paper, we give a complete investigation to real matrix representations of generalized quaternions, and consider their various applications to generalized quaternions as well as matrices of generalized quaternions. And then, Frenet- Serret formulas are studied with help of the generalized quaternions again by the method in [1].**

Keywords **– Generalized Quaternions, Frenet-Serret Formulas, Matrix Equation, Tangent, Normal, Binormal Unit Vector, Conjugate.**

I. Introduction

In 1843, Sir W.R. Hamilton introduce the quaternions as

$$H=\{a=a_0+a_1i+a_2j+a_3k\ ;\ w_{0-3},\alpha,\beta\in\mathbb{R}\}.$$

The natural extension of complex numbers, forming an associative algebra under the addition and the multiplication. This algebra is an effective way for understanding many aspects of physics and kinematics. Nowadays, quaternions are used especially in the area of computer vision, computer graphics, animation and to solve optimization problems involving the estimation of rigid body transformations too. Obtaining the roots of a quaternion was given by Niven, (1942) and Brand, (1942). A brief introduction of the generalized quaternions is provided also, this subject have investigated in algebra (see, [2], [3]). Recently, generalized quaternions are studied and given some of their algebraic properties.(see, [4]) The Frenet-Serret formulas for quaternions (see, [5], [6]) are described the kinematic properties of a particle which moves along a continuous, differentiable curve in Euclidean space or Minkowski space. These formulas have a common area of usage inmathematics, physics (especially in relative theory), medicine, computer graphics and such fields. Later, our major objective is to vent how the generalized quaternions formalism can be applied with great success not only to the interpolation between coordinate frames, but also to a remarkably description of the evolving coordinate-frame geometry of curves.

II. Generalized Quaternions

A generalized quaternion is represented as

$$H_{\alpha,\beta}=\{w=w_0+w_1I+w_2j+w_3k\ ;\ w_{0-3},\alpha,\beta\in\mathbb{R}\},$$

where i,j,k are special elements of $H_{\alpha,\beta}$ satisfying

$$i^2=-\alpha,\quad j^2=-\beta,\quad k^2=-\alpha\beta \tag{1}$$

$$ij=-ji,\quad jk=\beta i=-kj,\quad ki=\alpha j=-ik\ ;\ \alpha,\beta\in\mathbb{R}. \tag{2}$$

In that case, $H_{\alpha,\beta}$ is a algebra This algebra is called generalized quaternion algebra and are denoted by $H_{\alpha,\beta}$, One of the basis of this algebra is $\{1,i,j,k\}$. Addition of two generalized quaternions

$$w=w_0+w_1i+w_2j+w_3k$$

and

$$v=v_0+v_1i+v_2j+v_3k$$

is defined by

$$v+w=(v_0+w_0)+(v_1+w_1)i+(v_2+w_2)j+(v_3+w_3)k$$

whereas multiplication is defined by

$$\begin{aligned} v.w=&(v_0w_0-\alpha v_1w_1-\beta v_2w_2-\alpha\beta v_3w_3)+\\ &(v_0w_1+v_1w_0+\beta v_2w_3-\beta v_3w_2)i+\\ &(v_0w_2+v_2w_0+\alpha v_3w_1-\alpha v_1w_3)j+\\ &(v_0w_3+v_1w_2-v_2w_1+v_3w_0)k. \end{aligned} \tag{3}$$

Definition: (*The Concept of Conjugate*): For a given generalized quaternion w, where $\overline{w}$ denote the conjugate of generalized quaternion w,

$$\overline{w}=w_0-w_1i-w_2j-w_3k$$

This operation satisfies

$$\overline{v+w}=\bar{v}+\overline{w}$$

$$\overline{v.w}=\bar{v}.\overline{w}$$

for $v,w\in H_{\alpha,\beta}$

Definition (Norms of Generalized Quaternions): The norm of w is defined to be

$$|w|=\sqrt{w.\overline{w}}=\sqrt{(w_0^2+\alpha w_1^2+\beta w_2^2+\alpha\beta w_3^2)},$$

$$N_w=w_0^2+\alpha w_1^2+\beta w_2^2+\alpha\beta w_3^2. \tag{4}$$

Lemma: Although $H\alpha,\beta$ is nonassociative, it is still an alternative, flexible and division algebra over $\mathbb{R}$, that is, for all $v,w\in H\alpha,\beta$ the following equalities hold:

1. $v(v.w)=v^2.w$
2. $(vw).w=v.w^2$
3. $v(v.w)=(vw).w$
4. $|v.w|=|v||w|$
5. $w^{-1}=\frac{w}{w^2}$
6. $(Imw)^2=-|Imw|^2$

For the generalized quaternion algebra $H\alpha,\beta$. It is well known that through the bijective map [7];

$$\varphi: w\rightarrow\ \varphi(\omega)=\begin{bmatrix} w_0 & -\alpha w_1 & -\beta w_2 & -\alpha\beta\omega_3\\ w_1 & w_0 & -\beta w_3 & \beta w_2\\ w_2 & \alpha w_3 & w_0 & -\alpha w_1\\ w_3 & -w_2 & w_1 & w_0 \end{bmatrix} \tag{5}$$

$H\alpha,\beta$ is algebraically isomorphic to the matrix algebra

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$$M=\left\{\begin{bmatrix} w^0 & -\alpha w^1 & -\beta w^2 & -\alpha\beta w^3 \\ w^1 & w^0 & -\beta w^3 & \beta w^2 \\ w^2 & \alpha w^3 & w^0 & -\alpha w^1 \\ w^3 & -w^2 & w^1 & w^0 \end{bmatrix} : w^{0-3}, \alpha, \beta \in \mathbb{R}\right\} \quad (6)$$

and $\varphi(w)$ is a real matrix representation of w. Our consideration for matrix representation of the generalized quaternions are based on Eq(5) and the result in Eq(6).

Definition (Generalized Inner Product): A generalized inner product is defined in $\mathbb{R}^4$,

$\langle u,w\rangle = u_0 w_0 + \alpha u_1 w_1 + \beta u_2 w_2 + \alpha\beta u_3 w_3$

where,

$u = (u_0, u_1, u_2, u_3)$, $w = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$.

III. The Formulas of Generalized Quaternions Frames

In differential geometry, *the Frenet-Serret formulas* describe the kinematic properties of a particle which moves along a continuous, differentiable curve in three-dimensional Euclidean space R^3, or the geometric properties of the curve itself irrespective of any motion. More specifically, the formulas describe the derivatives of the so-called *tangent*, *normal*, and *binormal unit vectors* in terms of each other. The tangent, normal, and binormal unit vectors, often called V_1,V_2 and V_3 or collectively the Frenet--Serret frame or V_1,V_2 and V_3 frame, together form an orthonormal basis spanning R^3, and are defined as follows: V_1 is the unit vector tangent to the curve, pointing in the direction of motion. V_2 is the normal unit vector, the derivative of V_1 with respect to the arclength parameter of the curve, divided by its length. V_3 is the binormal unit vector, the cross product of V_1 and V_2. *The Frenet--Serret formulas* are

$$dV_1 = v\, k_1\, V_2$$

$$dV_2 = -v\, k_1\, V_1 + v\, k_2 V_3$$

$$dV_3 = -v\, k_2\, V_2$$

where $\upsilon = \|a'\|$ is the derivative with respect to arc-length, k_1 is the curvature and k_2 is the torsion of the curve. The two scalars k_1 and k_2 define the curvature and torsion of a space curve. The associated collection, V_1,V_2,V_3 ,k_1 and k_2 is called the *Frenet--Serret* apparatus.

The Frenet-Serret formulas for Generalized Quaternions: The Frenet--Serret formulas are defined as generalized quaternion

$w = w_0 + w_1 i + w_2 j + w_3 k$

and characterized by the following : The rotation B (a 3x3 orthogonal matrix) can be explained with two the different generalized quaternion. These are w and $-w$, using transform law,

$w\vec{u}\bar{w} = B.\vec{u}$

where $u=u_1 i+u_2 j+u_3 k$ is a generalized pure quaternion. We define B from Eq(5),

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad (7)$$

$b_{11}: w_0^2 + \alpha w_1^2 - \beta w_2^2 - \alpha\beta w_3^2$

$b_{12}: 2\beta\omega_1\omega_2 - 2\beta\omega_0\omega_3$

$b_{13}: 2\beta\omega_0\omega_2 + 2\alpha\beta\omega_1\omega_3$

$b_{21}: 2\alpha\omega_0\omega_3 - 2\alpha\omega_1\omega_2$

$b_{22}: w_0^2 - \alpha w_1^2 + \beta w_2^2 - \alpha\beta w_3^2$

$b_{23}: 2\alpha\beta\omega_2\omega_3 - 2\alpha\omega_0\omega_1$

$b_{31}: 2\alpha\omega_1\omega_3 - 2\omega_0\omega_2$

$b_{32}: 2\omega_0\omega_1 + 2\beta\omega_2\omega_3$

$b_{33}: w_0^2 - \alpha w_1^2 - \beta w_2^2 + \alpha\beta w_3^2$

All rows of B explained in this form are orthogonal but not orthonormal. We get

$$B' = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \end{bmatrix}$$

$$b'_{11}: \frac{1}{\alpha} w_0^2 + \alpha w_1^2 - \frac{\beta}{\alpha} w_2^2 - \beta w_3^2$$

$$b'_{12}: 2w_1 w_2 - 2w_0 w_3$$

$$b'_{13}: \frac{2}{\alpha} w_0 w_2 + 2w_1 w_3$$

$$b'_{21}: 2w_0 w_3 - 2w_1 w_2$$

$$b'_{22}: \frac{1}{\alpha} w_0^2 - \frac{1}{\beta} w_1^2 + w_2^2 - \alpha w_3^2$$

$$b_{'23}: 2w_2 w_3 - \frac{2}{\beta} w_0 w_1$$

$$b'_{31}: 2w_1 w_3 - \frac{2}{\alpha} w_0 w_2$$

$$b'_{32}: \frac{2}{\beta} w_0 w_1 + \beta w_2 w_3$$

$$b'_{33}: \frac{1}{\alpha\beta} w_0^2 - \frac{1}{\beta} w_1^2 - \frac{1}{\alpha} w_2^2 + w_3^2$$

All rows of B (B is rotation matrix) explained in this form are orthonormal and create a roof.

Let $\alpha=\beta=1$ be given. Then w is the real quaternion, and for $\alpha=1$, $\beta=-1$ the split quaternion. If the rows in eq(7) is derived respectively, then following results are obtained;

$$d\vec{V}_1 = 2[X][dw]$$
$$d\vec{V}_2 = 2[Y][dw] \quad (8)$$
$$d\vec{V}_3 = 2[Z][dw]$$

The Frenet equations themselves must take the form

$2[X][dw] = dV_1 = vk_1 V_2$

$2[Y][dw] = dV_2 = -vk_1 V_1 + \upsilon k_2 V_3$ (9)

$2[Z][dw] = dV_2 = -\upsilon k_2 V_2$

where,

$$[dw] = \begin{bmatrix} dw_0 \\ dw_1 \\ dw_2 \\ dw_3 \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (10)$$

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eventually, with Eq.(8),(9) and Eq.(10) the following equations are obtain:

$\rho_0=0 \quad \rho_1=-(\upsilon/2)\alpha k_2 \quad \rho_2=0 \quad \rho_3=-(\upsilon/2)\alpha k_1$

$\gamma_0=(\upsilon/2)(k_2)/\alpha \quad \gamma_1=0 \quad \gamma_2=(\upsilon/2)k_1 \quad \gamma_3=0$

$\eta_0=0 \quad \eta_1=-(\upsilon/2)\,(k_2)/\beta \quad \eta_2=0 \quad \eta_3=(\upsilon/2)\alpha k_2$

$\mu_0=(\upsilon/2)(k_1)/\beta \quad \mu_1=0 \quad \mu_2=-(\upsilon/2)(k_2)/\alpha \quad \mu_3=0$

Thus, the generalized quaternion Frenet frame eqution;

$$[dw]=\begin{bmatrix} dw_0 \\ dw_1 \\ dw_2 \\ dw_3 \end{bmatrix} = (\upsilon/2)\begin{bmatrix} 0 & -\alpha k_2 & 0 & -\beta k_1 \\ \frac{k_2}{\alpha} & 0 & k_1 & 0 \\ 0 & -\frac{k_1}{\alpha} & 0 & \alpha k_2 \\ \frac{k_1}{\beta} & 0 & -\frac{k_2}{\alpha} & 0 \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$2V_0' = -\alpha k_2 V_1 - \alpha k_1 V_3$

$2V_1' = \frac{k_2}{\alpha} V_0 + k_1 V_2$

$2V_2' = -\frac{k_1}{\beta} V_1 + \alpha k_2 V_3$

$2V_3' = \frac{k_1}{\beta} V_0 - \frac{k_2}{\alpha} V_2.$

$$\begin{bmatrix} V_0' \\ V_1' \\ V2' \\ V_3' \end{bmatrix} = (\upsilon/2)\begin{bmatrix} 0 & -\alpha k_2 & 0 & -\beta k_1 \\ \frac{k_2}{\alpha} & 0 & k_1 & 0 \\ 0 & -\frac{k_1}{\beta} & 0 & \alpha k_2 \\ \frac{k_1}{\beta} & 0 & -\frac{k_2}{\alpha} & 0 \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

2000 Mathematics Subject Classification. 11R52, 15A24, 15A90, 53A17.

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AUTHOR'S PROFILE

Faik Babadağ
received B.Sc. degre from Ankara University , Ankara, Turkey in 1991, Ph. D. degres in Mathematics from Ankara University, Ankara, Turkey in 2007, respectively. He is an Associate Professor in the department of Mathematics at Kırıkkale University, Kırıkkale, Turkey. His research interests are differential geometry and matrix theory.

Address: Kırıkkale University, Art & ScienceFaculty Kırıkkale, Turkey.
Email: faik.babadag@kku.edu.tr

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