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Diffusion in Slightly Varying Non-Uniform Dialyzer

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***Abstract* – In this paper, we investigate the variation of concentration of urea in a viscous, incompressible, laminar and Newtonian fluid that flows in a divergent/convergent tube of slightly varying non-uniform cross-section with semi-permeable wall. The concentration in the fluid is considered as a function of both axial and radial distances. The governing diffusion equation is solved analytically. The effects of slope parameter k, amplitude ratio ε (ratio of amplitude to the inlet width) and Sherwood number on the concentration of urea in the blood are presented graphically. The results shows that the variation of slope parameter, amplitude ratio and Sherwood number have considerable influences on the concentration level in such a fluid flow.**

***Keywords* – Concentration, Dialyser, Permeable Wall.**

I. Introduction

It is an established fact that the main function of kidneys is to maintain the chemical quality of blood by removing the waste products contributed to the blood stream by the metabolic processes in the human body. In particular, kidneys help to remove urea. When they malfunction it results in many severe complications which may prove to be fatal. In such cases one of the alternatives is to take the impure blood out of the body, remove urea from it, and then return the purified blood to the body. This process is accomplished using a device called hemodialyser. A dialyzer is an artificial kidney designed to provide controllable transfer of solutes and water across a semi-permeable membrane separating flowing blood and dialysate streams. A dialysate is usually a solution of some chemicals in water which contains many of the important substances found in blood and does not contain urea, uric acid or other wastes in blood. The semi-permeable membrane can be a cellulose membrane that permits a metabolic product, say, urea, to pass through it. Inside one of the channels of the dialyzer, blood is made to flow and outside of which dialysate flows. During this flow, wastes that have accumulated in the blood will diffuse across the membrane into the dialysate, in which the urea and other wastes concentration is maintained lower than that of the blood by maintaining a continuous supply of fresh dialysate to it. Hence, a dialyzer enables blood to get an external bath so that waste products will diffuse through the membrane due concentration difference and deposit on the outside of the membrane to be removed as a form of urine by the flow of the dialysate.

Flow in dialyzers has been studied by many authors. Kapur (1981), solved the conjugate boundary value problem of mass transfer in dialysers by applying Galerkin's method. The problem by Kapur (1981) was an extension of the problem by Cooney et al.(1974). Cooney et al solved the problem for the flat plate dialyser while Kapur solved for circular duct dialyser. Again Kapur (1982), discussed oxygen concentration profiles in capillaries and living tissues for general linear kinetics along with axial diffusion. He (1982) solved the governing equations for the tissue and capillary regions which are coupled by the boundary conditions at the capillary wall by Galerkin's method considering a constant cross sectional cylindrical duct of Krogh (1919).

Kolev et al (1992), derived a relationship for predicting the mass transfer coefficient/Sherwood number and the Peclet number in a parallel-plate laminar flow system with one impermeable wall and an opposite wall at which the concentration is uniform. They discussed mass transfer under laminar flow on the basis of the Navier-Stokes equations and the axially dispersed plug-flow model. Later they also determined the absolute limits of mass transfer across the membrane in a parallel-plate dialyser set by the flow pattern in both channels on the basis of a mathematical model assuming axially dispersed plug flow.

Ho-Ming Yeh (2011), investigated the effect of internal recycle on the mass transfer rate in counter currently parallel-flow dialysers for improved performance. He achieved a considerable improvement in performance, in contrast to the single-pass device of same size without recycle, by assuming one of the flow channels as divided in to two sub channels of same width with one performed as the operating sub channel and the other as the reflux cochannel, which provides the increase in the fluid velocity, resulting in reduction of mass transfer resistance.

All the above investigators discussed the mass transfer in a dialyser by considering either uniform cross-sectional circular duct or parallel plate dialyser. However, in such investigations it is also important to see the effect of the wavy boundaries on the concentration of urea as the blood flows through dialyser.

Hence, in the present study, an attempt is made to investigate the variation of concentration of urea due to diffusion when the blood passes through converging/diverging slightly varying non-uniform cross-sectional tube dialyser. The flow is considered as incompressible, laminar and steady. This type of study is important as the concentration in the uniform dialyser can be a particular case of the present problem of slightly varying non-uniform cross-sectional dialyser.

The boundary of the tube is assumed to be symmetric about z axis and it is given as in [10]:

$$\eta(z) = R_0 + k_1 z + a\sin(\frac{2\pi z}{\lambda}) \qquad (1)$$

where R_0 is the radius of the tube at the inlet $(z=0)$, k_1 is a constant whose magnitude depends on the length of the tube exit and inlet dimension and is assumed to be

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$<<1$, a is the amplitude and λ is the wave length, as shown in Figure (1).

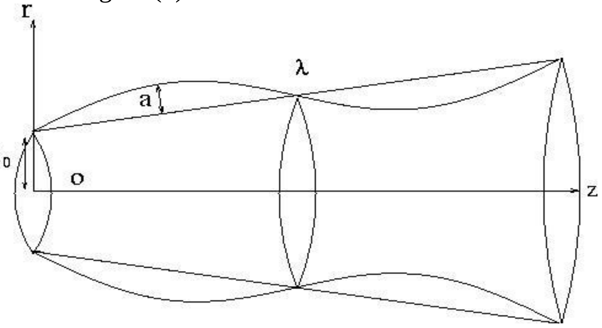

Fig.1. Flow Geometry

II. Mathematical Formulation

Consider an incompressible steady Newtonian fluid flow through a converging/diverging slightly varying non-uniform cross-sectional tube. Neglecting axial diffusion (assuming large Peclets numbers), the concentration of urea in the flow is governed by the diffusion convection equation [1]:

$$D\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}\right)=V(r)\frac{\partial C}{\partial z}=Vm\left(1-\frac{r^2}{R_0^{\ 2}}\right)\frac{\partial C}{\partial z}, \tag{2}$$

where $C(r,z)$ is the concentration of urea in the blood, D is the coefficient of diffusion, $V(r)$ is the velocity distribution inside the tube and V_m is the maximum velocity.

To set the boundary conditions we assume the blood flowing in to the dialyser contains a constant concentration, say "C_{in}" at the inlet and a continuous constant flow of dialysate in to the dialyser, say "C_d".

The boundary conditions are:

$$\frac{\partial C}{\partial r}=0,\ \text{at } r=0. \tag{3}$$

$$C(r,z)=C_{in},\ \text{at } z=0,\ 0\le r\le R_0 \tag{4}$$

$$-D\frac{\partial C(r,z)}{\partial r}=P\left(C(r,z)-C_d\right),\ at\ r=\eta(z), z>0 \tag{5}$$

where P is the permeability of the membrane separating the blood and dialysate in the dialyser.

Equation (3) arises from consideration of flow symmetry or from the fact that concentration is maximum on the axis. Equation (4) refers the assumption of constant entry of concentration at $z=0$. Equation (5) expresses the fact that the rate of concentration from blood to the dialysate across the membrane is proportional to the product of the permeability P of the membrane and the difference of concentrations on the two sides.

Using the following non-dimensional quantities

$$C'=\frac{C-C_d}{C_{in}-C_d}, \qquad r'=\frac{r}{R_0},$$

$$z'=\frac{z}{R_0 P_e}, \qquad \eta'=\frac{\eta}{R_0},$$

where, P_e refers Peclets number, the governing equation (2) transformed in to the non-dimensional form as (after dropping the primes)

$$\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}=\left(1-r^2\right)\frac{\partial C}{\partial z}, \tag{6}$$

Similarly, the boundary conditions (3), (4) and (5) transformed to:

$$\frac{\partial C}{\partial r}=0,\ \text{at } r=0. \tag{7}$$

$$C(r,z)=1,\ \text{at } z=0,\ 0\le r\le 1 \tag{8}$$

$$\frac{\partial C(r,z)}{\partial r}+ShwC(r,z)=0,$$
$$at\ r=\eta(z)=1+kz+\varepsilon\sin(2\pi z), z>0 \tag{9}$$

where, $Shw=\dfrac{R_0 P}{D}$ is Sherwood number, $k=\dfrac{k_1\lambda}{R_0}$ is slope parameter and $\varepsilon=\dfrac{a}{R_0}$ is amplitude ratio of the wavy walls.

To solve equation (6) together with the corresponding conditions (7), (8) and (9) we use analytical method by considering:

$$C(r,z)=U(r)W(z) \tag{10}$$

to be solution of (6) and in substituting we get the following system of differential equations:

$$\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}+\lambda_n^{\ 2}(1-r^2)U=0, \tag{11}$$

$$\frac{dW}{dz}+\lambda_n^{\ 2}W=0, \tag{12}$$

Where $-\lambda_n^{\ 2}$ is separation constant. Solving equations (11) and (12) the solution (10) gives:

$$C(r,z)=B_1\left(1-\frac{\lambda_n^{\ 2}}{4}r^2+\frac{\lambda_n^{\ 2}}{4^2}\left(1+\frac{\lambda_n^{\ 2}}{4}\right)r^4\right)e^{-\lambda_n^{\ 2}z}$$
$$+B_2\left(1-\frac{\lambda_n^{\ 2}}{4}r^2+\frac{\lambda_n^{\ 2}}{4^2}\left(1+\frac{\lambda_n^{\ 2}}{4}\right)r^4\right)Ge^{-\lambda_n^{\ 2}z} \tag{13}$$

Where

$$G=\frac{1}{\left(\frac{\lambda_n^{\ 2}}{4^2}\left(1+\frac{\lambda_n^{\ 2}}{4}\right)\right)^2}\left[\begin{array}{l}\ln\left(r^{A_1}(r-s)^{A_2}(r+s)^{A_3}(r-t)^{A_4}(r+t)^{A_5}\right)\\ \qquad -\left(\dfrac{A_6}{r-s}+\dfrac{A_7}{r+s}+\dfrac{A_8}{r-t}+\dfrac{A_9}{r+t}\right)\end{array}\right],$$

For

$$s=\sqrt{\frac{2}{1+\frac{\lambda_n^2}{4}}+\frac{4i}{\lambda_n\left(1+\frac{\lambda_n^2}{4}\right)}}$$ and

$$t=\sqrt{\frac{2}{1+\frac{\lambda_n^2}{4}}-\frac{4i}{\lambda_n\left(1+\frac{\lambda_n^2}{4}\right)}}$$

The constants A_1 to A_9 are determined through the conditions given in (7) and (8). Using the boundary conditions (7) and (8) we get $B_1=1$ and $B_2=0$. Hence, the solution of (6) becomes:

$$C(r,z)=\left(1-\frac{\lambda_n^2}{4}r^2+\frac{\lambda_n^2}{4^2}\left(1+\frac{\lambda_n^2}{4}\right)r^4\right)e^{-\lambda_n^2 z}.$$

Now, to determine the eigenvalues "λ_n" we use the boundary condition (9), then solving the resulting equation using Mathematica up to order four we get:

$$\lambda_1=-\sqrt{2}\sqrt{-\frac{A_1+\sqrt{A_2}}{A_3}},\ \lambda_2=\sqrt{2}\sqrt{-\frac{A_1+\sqrt{A_2}}{A_3}},$$

$$\lambda_3=-\sqrt{2}\sqrt{\frac{-A_1+\sqrt{A_2}}{A_3}},\lambda_4=\sqrt{2}\sqrt{\frac{-A_1+\sqrt{A_2}}{A_3}}$$

Where,

$$A_1=-8\eta+4\eta^3-4\eta^2 Shw+\eta^4 Shw,$$

$$A_2=\eta^2\left(\begin{array}{l}64-64\eta^2-48\eta^3 Shw+8\eta^5 Shw+\\ \qquad \eta^6 Shw^2-8\eta^4\left(-2+Shw^2\right)\end{array}\right),$$

$$A_3=\eta^3\left(4+\eta Shw\right).$$

III. Results and Discussion

The objective of this analysis is to see the effect of wavy walls of diverging/converging and slightly varying non-uniform cross-sectional tube dialyser on the concentration $C(r,z)$ as the blood passes through it. It may be recalled that k characterizes the slope of the diverging/converging wavy walls and ε refers the amplitude ratio of the wavy walls. $k=0$, represents a rigid tube of slightly varying non-uniform cross-sectional (sinusoidal) tube and $\varepsilon=0$, represents a rigid tube of slightly varying cross-sectional tube.

In what follows, we discuss the effect of these parameters together with Sherwood number on the concentration $C(r,z)$. The concentration $C(r,z)$ is obtained by taking different values of k for different values of ε and Sherwood number Shw. The values of k are taken as -0.1 for convergent tube, 0 for normal non-uniform (sinusoidal) tube and 0.1 for divergent tube respectively.

It can be observed from Figure (2) that the concentration $C(r,z)$ decreases rapidly for the uniform tube than for the divergent non-uniform tube up to $z=0.6$. But when $z>0.6$, $C(r,z)$ assume the same behaviour for both cases. It can also be noted that the concentration $C(r,z)$ reduces for the uniform tube than the convergent non-uniform tube up to $z=0.4$. However, after $z=0.4$, the concentration $C(r,z)$ has less value for non-uniform convergent tube than uniform tube,as seen in Figure (3). This may be due to the fact that the tube gets symmetrically narrowed down.

In Figure (4), the values of the concentration $C(r,z)$ over the length of the tube are calculated for different values of the slope parameter k . As observed, the concentration $C(r,z)$ has less values for the convergent tube than the normal (sinusoidal) and divergent tubes. This implies that the tube has to be significantly convergent to have less concentration.

In figure (5), the influence of amplitude ratio ε on concentration is presented. It can be seen that as ε increases from 0.07 to 0.13, the concentration $C(r,z)$ decreases slowly up to half of the length of the tube. After this point the reverse is true.

The effects of Sherwood number Shw on the concentration $C(r,z)$ of the fluid are presented in Figures (6)-(8). As it is seen that as the Sherwood number increases, the concentration $C(r,z)$ decreases for all forms of the tube (convergent, sinusoidal or divergent tubes). Hence, we conclude that there is no significant difference on the variation of the concentration $C(r,z)$ for these three different forms of the tube.

IV. Conclusion

The main purpose of this study is to investigate the variation of concentration $C(r,z)$ when the impure blood passes through a divergent/convergent tube of slightly varying non-uniform cross-sectional tube dialyser. From the results obtained, it is observed that the concentration $C(r,z)$ decreases rapidly for the uniform tube than for the divergent non-uniform tube up to an axial distance of $z=0.6$. However after this point the concentration $C(r,z)$ assumes the same value for both cases. It is important to mention that the tube has to be convergent so that we obtain less concentration at the end. Further, it is observed that the concentration has less value for the convergent tube than for the normal (sinusoidal) tubes and divergent tubes. Finally, we observe that as the Sherwood number increases the concentration $C(r,z)$ decreases for all forms of the tubes (Convergent, sinusoidal or divergent).

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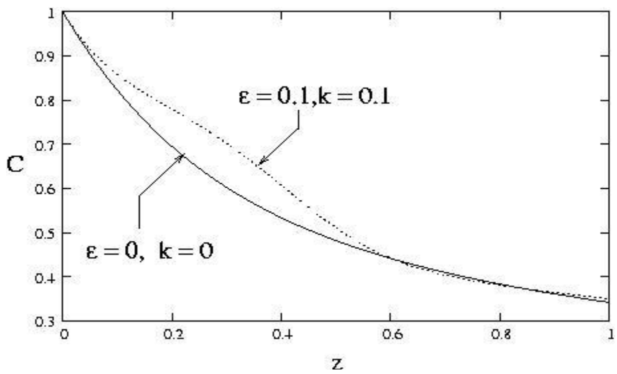

Fig.2. Variation of $C(r,z)$ with z, for $\varepsilon = 0.1, k = 0.1$ and $\varepsilon = 0, k = 0,$ keeping $Shw = 0.1.$

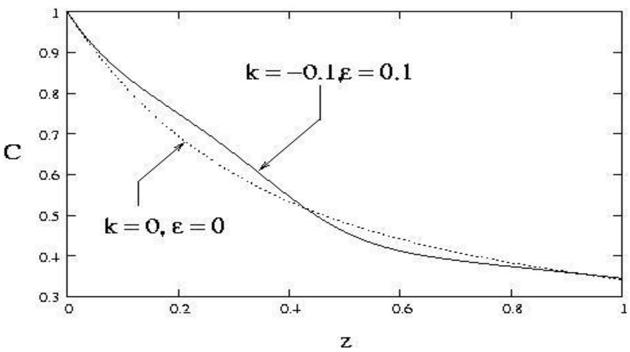

Fig.3. Variation of $C(r,z)$ with z, for $\varepsilon = 0.1, k = -0.1$ and $\varepsilon = 0, k = 0,$ keeping $Shw = 0.1.$

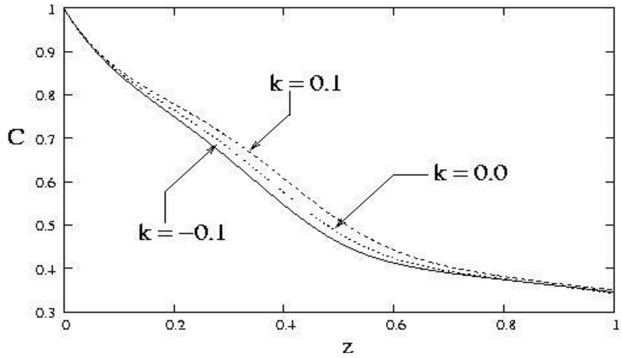

Fig.4. Variation of $C(r,z)$ with z due k keeping $\varepsilon = 0.1$ and $Shw = 0.1.$

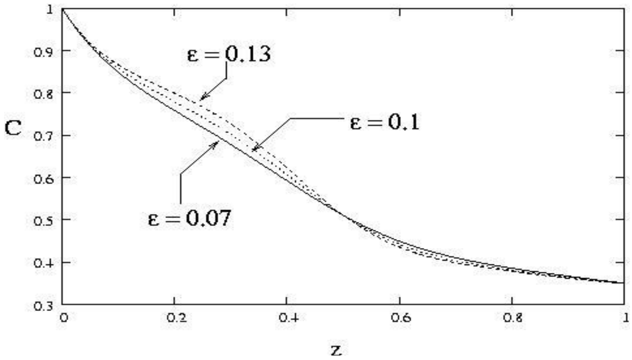

Fig.5. Variation of $C(r,z)$ with z due ε keeping $k = 0.1$ and $Shw = 0.1.$

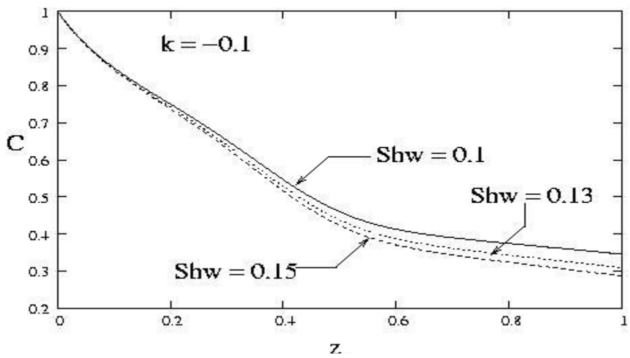

Fig.6. Variation of $C(r,z)$ with z due Shw keeping $\varepsilon = 0.1$ and $k = -0.1.$

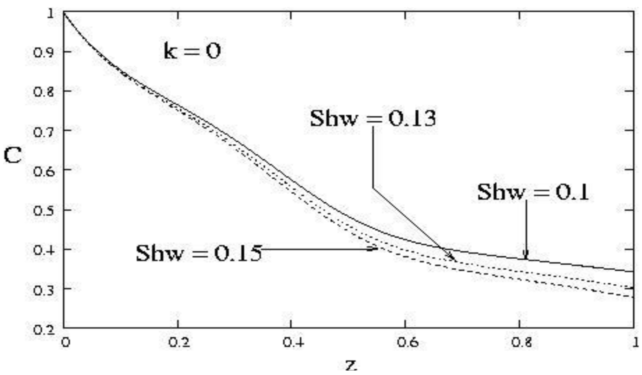

Fig.7. Variation of $C(r,z)$ with z due Shw keeping $\varepsilon = 0.1$ and $k = 0.$

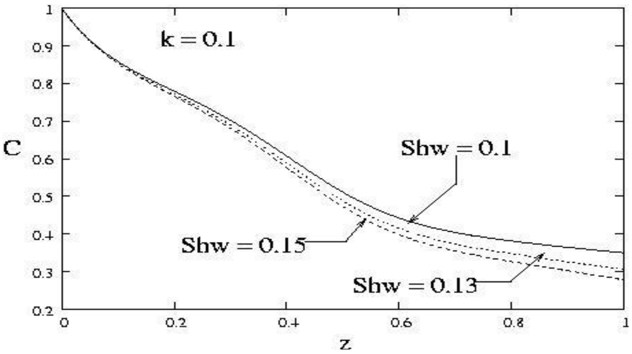

Fig.8. Variation of $C(r, z)$ with z due Shw keeping $\varepsilon = 0.1$ and $k = 0.1$.