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Subdividing Operation for Extension of Graphs

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Abstract – **In this paper, we introduced the concept of subdivided extension graphs. The subdividing extensible class of graphs are defined. Furthermore, the subdivided extensibility number are studied. The regular graphs, which have subdivided extensibility number 1, 2 or 3, are characterized.**

Keywords – **Regular Graph, Extension of Graphs, Eulerian Graphs Subdividing of Graph, Reducibility of Graphs.**

I. Introduction

Kharatand Waphare [2001] introduced the concept of reducibility number for posts in Lattices Theory. Akram [2005] introduced analogous concept in graph theory. In fact, he studied the reducibility number for some classes of graphs. Akram [2007] introduced the contractibility number for some classes of graphs. Attar [2009] introduced the concept of vertex extension graphs (digraphs). In fact he characterized the extensibility number for some graphs (digraphs). Akram and Ahmed [2013] introduced a new definition for extension graphs (digraphs). Further, they defined the extensible class of graphs (digraphs), and they characterized regular graphs (digraphs) which have extensibility number 1, 2 or 3. Akram and Ahmed [2014], characterized Eulerian graphs which have extensibility number 1, 2 or 3. In this work, we defined the subdividing extension of graphs. Further, the subdivided extensibility number has been introduced. Furthermore, the regular graphs, which have subdivided extensibility number 1, 2 or 3, have been characterized.

For the undefined concepts, we refer the reader for Clark, J. and Holton [1991], West, D. B [1999], Harary, F[1969]. All the graphs throughout this paper are simple.

II. Subdivided Extension Graphs

Definition 1: Let G be a nonempty (nonnull) simple graph with n vertices, $n \geq 3$. Given a set of edges $S = \{e_1, e_2, \dots, e_m\}$ of G. We can construct a new simple graph from G by subdividing the edges of S by the vertices in the set $X = \{x_1, x_2, \dots, x_m\}$ respectively, in such a way every vertex of X is adjacent to at least one vertex of G different from the end vertices of the subdivided edge. We denote the new graph by G_S. We say that G_S is obtained from G by subdividing extension edges. S is called subdividing extension set. In particular, if S consists of a single element e, then e is called subdividing extension edge and the graph denoted by G_e. The transition from G to G_S is called subdividing extension operation.

Definition 2: Let $\mathfrak{R}$ be the class of graphs with certain property. Then $\mathfrak{R}$ is called subdividing extensible if for every graph $G \in \mathfrak{R}$, either G is null graph or there exist a subdividing extension edge such that $G_e \in \mathfrak{R}$.

Example 1:

1. The class of connected graphs is subdividing extensible.
2. The class of trees is not subdividing extensible.

Definition 3: Let ξ be a class of graphs with certain property, and $G \in \xi$ be a non-trivial (non null) with $n \geq 3$ vertices. The subdividing extensibility number of G with respect to ξ is the smallest positive integer m, if exists, such that there exists a subdividing extension set S of cardinality m in such a way the graph $G_S \in \xi$. We write $m = \text{sub} - \text{ext}_{\xi}(G)$. If such a number does not exist for G, then we say that the corresponding subdividing extensibility number is ∞.

Lemma 1: Let $\mathfrak{I}$ the class of regular graphs, $R \in \mathfrak{I}$, then $sub - ext_{\mathfrak{I}}(R) > 1$.

Proof: Let R be an r −regular graph, and let $e = uv$ be an edge in R. Suppose that $e = uv$ is sub divided by the vertex x. Then by Definition 1, x is adjacent to at least one vertex of R different from the end vertices of $e = uv$. Then each of the end vertices of $e = uv$ has a degree r in the graph R_e and there exists at least one vertex of R with degree $r + 1$. That is R_e is not regular.
Hence $sub - ext_{\mathfrak{I}}(R) > 1$. □

Theorem 1: Let $\mathfrak{I}$ be the class of regular graphs, $R \in \mathfrak{I}$ be an r-regular with even number of vertices $n \geq 4$. Then $sub - ext_{\mathfrak{I}}(R) = 2$ if and only if $r = \frac{n}{2} + 1$ and there exist two edges $e_1 = u_1v_1, e_2 = u_2v_2$ of R with no common end vertex such that each of the new vertices x, y which come from subdividing e_1,e_2 respectively is adjacent to $\frac{n}{2} + 2$ vertices of R and, $N(x) \cap N(y) = \{u_1, v_1, u_2, v_2\}$.

Proof: Let $\mathfrak{I}$ be the class of regular graphs, $R \in \mathfrak{I}$ be an r-regular with even number of vertices $n \geq 4$. Suppose that $sub - ext_{\mathfrak{I}}(R) = 2$. Then by Definition3, there exists a subdividing extension set $S = \{e_1, e_2\}$ of cardinality 2 such that $R_S \in \mathfrak{I}$ and S is the smallest such set.

Let $e_1 = u_1v_1$, $e_2 = u_2v_2$ and let x, y be the two vertices which are come from subdividing e_1, e_2 respectively. As R_S is simple regular then we have two cases:
(i) R_S is $(r + 1)$ −regular. Or
(ii) R_S is $(r + 2)$ −regular.

Suppose that (i) holds. That is R_S is $(r + 1) -$ regular graph. If $e_1 = u_1v_1$, $e_2 = u_2v_2$ have a common end vertex, say $v_1 = u_2$. Then $d(v_1) = d(u_2) = r$, in the graph R_S a contradiction to our assumption. Hence e_1, e_2 have no common end vertex. Assume that w is a vertex of R_S different from the end vertices of e_1, e_2 such that $w \in N(x) \cap N(y)$ then $d(w) = r + 2$ in R_S a contradiction. Hence $N(x) \cap N(y) = \{u_1, v_1, u_2, v_2\}$. Now, as R_S is $(r + 1)$ −regular, then every vertex of R is adjacent by exactly one vertex from the new two vertices x, y. As n is even and R_S is regular , then each of x, y is adjacent to $\frac{n}{2}$ vertices of R. But x is adjacent to the end vertices of e_1. Then $d(x) = \frac{n}{2} + 2$ in R_S. Similarly $d(y) = \frac{n}{2} + 2$ in R_S. As R_S is $(r+1)$ −regular. Then $r + 1 = \frac{n}{2} + 2 \Rightarrow r = \frac{n}{2} + 1$.

Suppose that (ii) holds. That is R_S is $(r+2)$ −regular. Then every vertex of R is adjacent by both x and y. But the end vertices of e_1 is already adjacent by x and R_S is simple, then $d(u_1) = d(v_1) = r + 1$ in R_S. Similarly for the end vertices of e_2, $d(u_2) = d(v_2) = r + 1$ in R_S a contradiction. Thus this case is impossible.

Conversely, suppose that there exist two edges $e_1 = u_1v_1, e_2 = u_2v_2$ of R with no common end vertex such that each of the new vertices x, y which come from subdividing e_1, e_2 respectively is adjacent to $\frac{n}{2} + 2$ vertices of R, $N(x) \cap N(y) = \{u_1, v_1, u_2, v_2\}$ and $r = \frac{n}{2} + 1$. Then the degree of x is $\frac{n}{2} + 2$, Similarly for $d(y) = \frac{n}{2} + 2$. As $r = \frac{n}{2} + 1$, then the degree of each vertex which is adjacent by x is $\frac{n}{2} + 1 + 1 = \frac{n}{2} + 2$, similarly for the vertices which are adjacent by y. Thus every vertex of R_S has a degree $\frac{n}{2} + 2$. Thus R_S is regular, and $\{e_1, e_2\}$ is subdividing extension set of cardinality 2. Hence $sub - ext_{\mathfrak{I}}(R) \le 2$. By lemma 1,
$sub - ext_{\mathfrak{I}}(R) > 1$.
Hence $sub - ext_{\mathfrak{I}}(R) = 2$□

Theorem 2: Let $\mathfrak{I}$ be the class of regular graphs, $R \in \mathfrak{I}$ be an r −regular with $n \ge 3$ vertices, $|E(R)| \ge 3$. Then $sub - ext_{\mathfrak{I}}(R) = 3$ if and only if n divisibleby 3 and R contain three edges $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3$ subdivided by the vertices x, y, z respectively, and one of the following holds.

1. $r = \frac{n}{3} + 1$, and every vertex of x, y, z is adjacent to $\frac{n}{3} + 2$ vertices of R such that each end vertex of e_1, e_2, e_3 is a neighbor to exactly two vertices from x, y, z and each vertex of R wich is different from the end vertices of e_1, e_2, e_3 is a neighbor to exactly one vertex from x, y, z.
2. $r = \frac{2n}{3}$, and every vertex of x, y, z is adjacent to $\frac{2n}{3} + 2$ vertices of R such that $N(x) \cap N(y) \cap N(z) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$, and each vertex in R which is different from the end vertices of e_1, e_2, e_3 is adjacent by exactly two vertices from x, y, z.
3. $r = \frac{n}{3} + 1$, and every vertex of x, y, z is adjacent to $\frac{n}{3} + 2$ vertices of R such that $N(x) \cap N(y) \cap N(z) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$ and every vertex of R which is different from the end vertices of e_1, e_2, e_3 is a neighbor to exactly one vertex of x, y, z.

Proof : Let $\mathfrak{I}$ be the class of regular graphs, $R \in \mathfrak{I}$ be an $r - regular$, $n \ge 3$ vertices, $|E(R)| \ge 3$. Suppose that $sub - ext_{\mathfrak{I}}(R) = 3$. Then by Definition3, there exists a set of three edges $S = \{e_1, e_2, e_3\}$ in R which are subdivided by the vertices x, y, z respectively such that $R_S \in \mathfrak{I}$ and S is the smallest such set. As we have three vertices x, y, z subdivided the edges e_1, e_2, e_3 respectively, and $d(x) = d(y) = d(z) in R_S$. Then $|N(x)| = |N(y)| = |N(z)|$. Thus n must be devisable by 3. Let $e_1 = u_1v_1, e_2 = u_2v_2, e_3 = u_3v_3$. Since e_1, e_2, e_3 are edges in R, then the induced subgraph induced by S in R is isomorphic to one of the following graphs: $P_1 \cup P_1 \cup P_1, C_3, P_2 \cup P_1, P_3, S_4$. Now, we discuss those cases:

Suppose that $[S]$ is isomorphic to $P_1 \cup P_1 \cup P_1$. That is e_1, e_2, e_3 are independent edges in R. As R_S is simple regular, then we have the following cases:
$(i) R_S\ is\ (r+1)$ −regular graph.
$(ii) R_S\ is\ (r+2)$ −regular graph. Or
$(iii) R_S\ is\ (r+3)$ −regular graph.

Suppose that (i) holds. That is $R_S is (r+1)$ −regular graph. That means the degree of every vertex of R increased by 1 in R_S. As such x must be adjacent to $\frac{n}{3}$ vertices other than u_1, v_1 that is $d(x) = \frac{n}{3} + 2$. Similarly for y and z. As $d(x) = d(y) = d(z)$ in R_S, then the end vertices of e_1, e_2, e_3 increased by 1 in R_S. Then u_1 is adjacent by one vertex from y, z. But u_1 is adjacent by x (x is a vertex subdivided e_1). Then u is a neighbor for exactly two vertices from x, y, z. Similarly for the other end vertices of e_1, e_2, e_3. For the vertices of R which are different from the end vertices of e_1, e_2, e_3. Since R_S is $(r+1)$ −regular, then each of those vertices must be a neighbor for exactly one vertex from x, y, z. Also as R_S is $(r+1)$ −regular and the degree of every vertex of x, y, z is $\frac{n}{3} + 2$, then $r + 1 = \frac{n}{3} + 2 \Rightarrow r = \frac{n}{3} + 1$.

Now, suppose that (ii) holds. That is R_S is $(r+2)-$ regular graph. That is the degree of every vertex of R increased by 2 in R_S. Since we have three vertices x, y, z subdivided the edges e_1, e_2, e_3 respectively and $d(x) = d(y) = d(z)$ in R_S. Thatis $|N(x)| = |N(y)| = |N(z)|$. Then n must be divisible by 3. Since R_S is $(r + 2$−regular. Then each of x,y,z is adjacent to $2n3$ vertices of R. But x is adjacent to u_1, v_1 (x subdivided e_1), then $d(x) = \frac{2n}{3} + 2$. Similarly for y and z. Thus $d(x) = d(y) = d(z) = \frac{2n}{3} + 2$. Now, as the degree of every vertex of the end vertices of e_1, e_1, e_3 increased by 2.Then u_1 is adjacent by y and z, but u_1 is adjacent by x also.Thus $u_1 \in N(x) \cap N(y) \cap N(z)$. Similarly for each of the others end vertices of e_1, e_2, e_3. Thus $\{u_1, v_1, u_2, v_2, u_3, v_3\} \in N(x) \cap N(y) \cap N(z)$. As R_S, is $(r+2) -$ regular, then every vertex of R which is different from the end vertices of e_1, e_2, e_3 is adjacent by exactly two vertices from x, y, z otherwise we get a contradiction to our assumption that R_S is $(r+2)$ −regular. Hence $N(x) \cap N(y) \cap N(z) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$.

As R_S, is $(r+2) -$ regular and $d(x) = d(y) = d(z) = \frac{2n}{3} + 2$, then $r + 2 = \frac{2n}{3} + 2 \Rightarrow r = \frac{2n}{3}$.

Now, suppose that (iii) holds. That is R_S, is $(r+3)$ −regular graph. That is the degree of every vertex of R increased by 3 in R_S. Since we have 6 end vertices of e_1, e_2, e_3 and the degree of each of them is $r + 3\ in\ R_S$, and we have 3 subdivided vertices x, y, z. Then each of the vertices x, y, z is adjacent to $\frac{3*6}{3}$ plus the end vertices of the subdivided edge. Thus each of x, y, z is adjacent to $\frac{3*6}{3} + 2 = 8$ from the 6 end vertices of e_1, e_2, e_3 , a

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contradiction for R_S is simple (Definition1). Hence this case is imposable.

Suppose that [S] is isomorphic to C_3. That is [S] forms a C_3 in R_S. In this case we have three end vertices fore_1, e_2, e_3.That is $u_1 = v_3, u_2 = v_1, u_3 = v_2$ andeach of those end vertices is a common end vertex of two edges frome_1, e_2, e_3.As R_S is a simple,the only possible case forR_S is $(r+1)$ −regular. In this case every vertex of R is adjacent by exactly one vertex from x, y, z .But each of the end vertices is $u_1 = v_3, u_2 = v_1, u_3 = v_2$ is a common neighbor for two of the edges e_1, e_2, e_3. Thus $N(x) \cap N(y) \cap N(z) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$.As $d(x) = d(y) = d(z)$inR_S, then each of x, y, z is adjacent to $\frac{n}{3}$ verticesof R,but each of x, y, z is adjacent to the end vertices of the subdivided edge ofe_1, e_2, e_3. Hence each of them is adjacent to$\frac{n}{3}+2$. AsR_S is regular, then$\frac{n}{3}+2 = r+1 \Rightarrow r = \frac{n}{3}+1$. If there exists a vertex in R which is different from the end vertices of e_1, e_2, e_3 is adjacent by two or three vertices from x, y, z, then we get a contradiction to $R_S$is simple$(r+1)$ −regular.

Suppose that [S] is isomorphic to $p_2 \cup p_1$. In this case two of the edges e_1, e_2, e_3 have a common end vertex , then the only case isR_S is$(r+1)$ −regular graph. As we have 5 end vertices for e_1, e_2, e_3 , and the regularity degree of R_S is$(r+1)$, so each of the 5 end vertices is adjacent by one vertex from x, y, z is impossible, so $R_S$is not regular a contradiction. By similar argument the cases P_3 and $S_4$are impossible.

Conversely, suppose that $n$divisibleby 3 and R contain three edges $e_1 = u_1v_1, e_2 = u_2v_2, and\ e_3 = u_3v_3$, which are subdivided by the vertices x, y, z respectively and one of 1 , 2 or 3 above is hold.

Suppose that 1 holds. That is $r = \frac{n}{3}+1$, and every vertex of x, y, z is adjacent to $\frac{n}{3}+2$ vertices of R such that each end vertex of e_1, e_2, e_3 is a neighbor to exactly two vertices from x, y, z and each vertex of R which is different from the end vertices of e_1, e_2, e_3 is a neighbor to exactly one vertex from x, y, z. In this case the set of edges $S = \{e_1, e_2, e_3\}$ is subdividing extension set of vertices, and it is clear that [S] forms a set of 3 independent edges, and$R_S$is regular. Hence $sub - ext(R) \le 3$. Suppose that $sub - ext(R) = 2$. Then by theorem1, R contain two edges with no common end vertex such that each of the two vertices which come from subdividing e_1, e_2 is adjacent to $\frac{n}{2}+2$ vertices of R, $N(x) \cap N(y) = \{u_1, v_1, u_2, v_2\}$ and $r = \frac{n}{2}+1$. But in this case n is even, and $r = \frac{n}{2}+1$. Thus $\frac{n}{2}+1 = \frac{n}{3}+1$, a contradiction. Hence $sub - ext_{\mathfrak{J}}(R) \ne 2$. By lemma1, $sub - ext_{\mathfrak{J}}(R) \ne 1$. Hence $sub - ext_{\mathfrak{J}}(R) = 3$.

Suppose that2 holds. That is $r = \frac{2n}{3}$, and every vertex of x, y, z is adjacent to $\frac{2n}{3}+2$ vertices of R such that $N(x) \cap N(y) \cap N(z) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$, and each vertex in R which is different from the end vertices of e_1, e_2, e_3 is adjacent by exactly two vertices from x, y, z. In this case it is easy to see that the set of edges $S = \{e_1, e_2, e_3\}$ is subdivided extension set of edges and [S] is a set of three independent edges, and R_S is regular. Hence $sub - ext_{\mathfrak{J}}(R) \le 3$. By using similar argument to 1 above, we get $sub - ext_{\mathfrak{J}}(R) \ne 2$, and by lemma1, $sub - ext_{\mathfrak{J}}(R) \ne 1$. Hence $sub - ext_{\mathfrak{J}}(R) = 3$.

Suppose that3 holds. That is $r = \frac{n}{3}+1$, and every vertex of x, y, z is adjacent to $\frac{n}{3}+2$ vertices of R such that $N(x) \cap N(y) \cap N(z) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$ and every vertex of R which is different from the end vertices of e_1, e_2, e_3 is a neighbor to exactly one vertex of x, y, z. In this case the set of edges $S = \{e_1, e_2, e_3\}$ is subdivided extension set, and [S] forms a 3-cycle. From assumption R_S is $(r+1)$ −regular graph. Hence $sub - ext_{\mathfrak{J}}(R) \le 3$. By similar argument to that in 1 above we get $sub - ext_{\mathfrak{J}}(R) \ne 2$ and by lemma1, $sub - ext_{\mathfrak{J}}(R) \ne 1$. Hence $sub - ext_{\mathfrak{J}}(R) = 3$, and the proof is complete.□

To illustrate Theorem3, we give the following example in Figure 1. In this Figure we see the order of graph and the regularity degree of this graph and the subdivided edges.

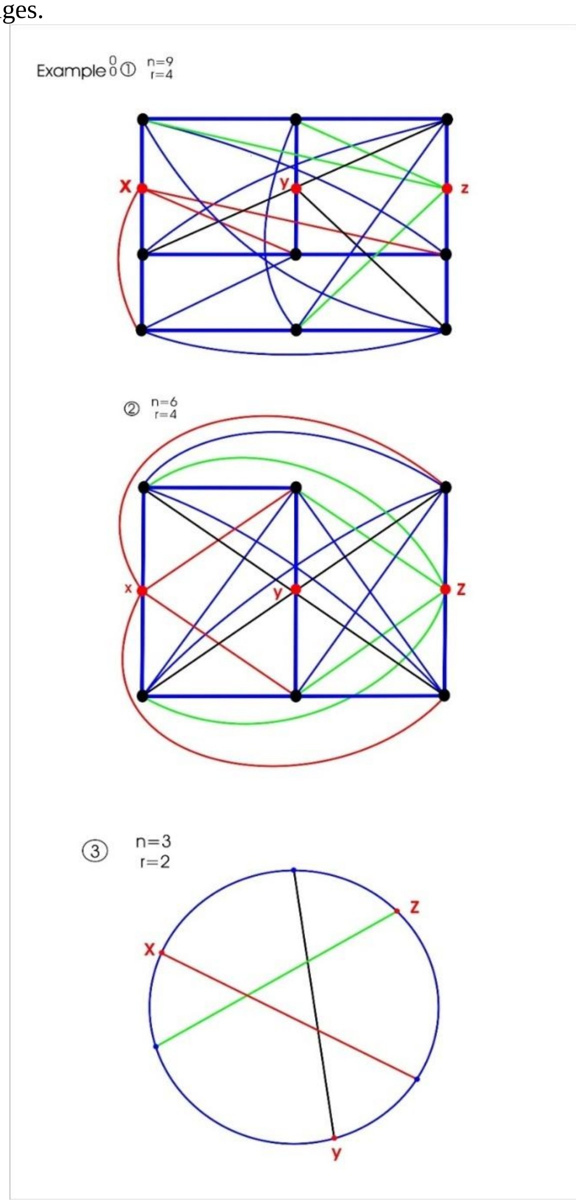

Fig.1

III. CONCLUSION

We conclude from this work that the graph can extended by subdividing its edges. That is we can get a graph with same property and extended number of vertices.

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