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# Exact Solutions of the Diffusion-Reaction Equation for a Coupled Shifted Harmonic Oscillator

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***Abstract* – We investigate the quasi-exact solutions of the diffusion-reaction equation for two dimensional non-hermitian Hamiltonian systems with in the frame work of extended complex phase space characterized by $x = x_1 + p_3$ , $p_x = p_1 + x_3$ , $y = x_2 + p_4$ , $p_y = p_2 + x_4$ . Analyticity property of the complex Hamiltonian system is exploited to obtain the explicit expressions for eigenvalues and eigenfunctions for a complex shifted harmonic potential and its PT-symmetric version.**

***Keywords* – D-R equation, Complex Hamiltonian, Eigenvalues and Eigenfunctions.**
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## I. Introduction

Recently, non-hermitian Hamiltonian systems have generated a lot of interest for the better theoretical understanding of some newly discovered phenomena in the area of physical sciences. But such studies in mathematical terms have not been reached to the desired level for the complex potentials. The complex potentials are used in the optical model of nucleus and some newly discovered phenomena in physics and chemistry, like the phenomena pertaining to resonance scattering in atomic, molecular, and nuclear physics and to some chemical reactions [1],[2]. A complex Hamiltonian H can provide real and bounded eigenvalues for certain domains of the underlying parameters if H is invariant under the simultaneous action of the PT reversal [3] .Therefore, it is now possible to study the complex Hamiltonians (PT-symmetric) which were not considered earlier for not meeting the hermiticity requirement[3]-[8]. The parity operator $\hat{p}$ and the time reversal operator $\hat{T}$ are defined by the action of position and momentum operator as

$P: x, y, p_x, p_{y,} i \rightarrow\ -x, -y, -p_x, -p_{y,}, i$

$T: x, y, p_x, p_{y,} i \rightarrow\ x, y, -p_x, -p_{y,}, -i$

The combined PT- operation becomes

$PT: x, y, p_x, p_{y,} i \rightarrow -x, -y,\ p_x, p_{y,}, -i$

The non-Hermitian PT-symmetric Hamiltonians have many applications in various fields of physics like superconductivity, population biology, quantum cosmology, condensed matter physics, quantum field theory, and so on [8].There are various ways of complexifying a given Hamiltonian [9] but here we use the scheme due to Xavier and de Aguir [10] used to develop an algorithm for the computation of the semiclassical coherent state propagator, to transform potentials in an extended complex phase space. In this approach, the transformations for the positions and the momenta variables are defined as

$$x = x_1 + ip_3\ ,\ p_x = p_1 + ix_3$$
$$y = x_2 + ip_4\ ,\ p_y = p_2 + ix_4 \qquad (1)$$

The presence of variables $(x_3\ , x_4\ , p_3\ , p_4\ )$ in the above transformations may be regarded as some sort of coordinate-momentum interactions of a dynamical system. Note that in this complexifying scheme, the degrees of freedom of the underlying system just become double.

It is well known that exact solutions of the Diffusion-Reaction (D-R) equation are possible for selected potentials and therefore, approximation methods are employed to obtain the solutions [11]-[13]. Besides the equation of continuity, KDV equation and diffusion reaction(D-R) equation and its variants are in a privileged class of differential equation in mathematics which have a wide application not only in physical sciences but in biological, social and economic sciences also. The presence of velocity term, which is also attributed to the turbulent or anomalous diffusion in the lateral version is found to play an important role in several applications particularly in biological studies and same is also responsible for making the corresponding Hamiltonian as non-hermitian [14]-[16]. Transformations similar to (1) have also been used in the study of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma[17]. Recently, in some studies the solutions of the D-R equation have been reported using the extended complex phase space approach [18]. However, these studies are confined to one dimensional systems only. An extension of such studies in higher dimensions is desirable to explore the possibilities of finding more applications. With this motivation, we have generalized extended complex phase space approach in two dimensions and studied some interesting complex systems and computed their eigenvalues and eigenfunctions. With the same spirit, in the present work, to expand the domain of applications, we investigate the solution of the D-R equation for a coupled shifted harmonic potential and its PT-symmetric version.

The organization of the paper is as follows: in section II, we develop the mathematical formulation of the extended complex phase space approach in two dimensions, for computing eigenvalue spectra of complex systems. In section III, using the results of section II, eigenvalues and eigenfunctions for the shifted harmonic potential are investigated. Finally, concluding remarks are presented in section IV.

## II. The Method

In this section we develop the mathematical formulation to calculate the eigenvalues and eigenfunction of the D-R

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equations for a given system. The linear version of the D-R is written as

$$-D\nabla^2 C(x,y,t) + v\nabla C(x,y,t) + u(x,y)C(x,y,t) = -\frac{\partial C(x,y,t)}{\partial t} \quad (2)$$

Also (2) can be compared with the time dependent SE (for ħ = m =1) as

$$-\frac{1}{2}\nabla^2\psi(x,y,t) + V(x,y)\psi(x,y,t) = i\frac{\partial\psi(x,y,t)}{\partial t} \quad (3)$$

Where, D-diffusion coefficient, v-velocity which is generally a function of x and t. But here v is constant and independent on both space and time. As t→∞ the complete solution of (2) vanishes where as solution of (3) remain periodic in time . Then (2) can be written as

$$C_t + v(C_x + C_y) = (C_{xx} + C_{yy}) - V(x,y)C(x,y) \quad (4)$$

Expressing (4) in the form

$$HC(x,y,,t) = -\frac{\partial C(x,y,t)}{\partial t} \quad (5)$$

Where H is the non-hermitian Hamiltonian operator given by

$$H = -D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + v\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + V(x,y) \quad (6)$$

The solution of (4) may be written as

$$C(x,y,t) = \psi(x,y)\tau(t)$$

As $H\psi(x,y) = \lambda\psi(x,y)$ (7)

Under the transformation (1), we derive

$$\frac{d}{dx} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}\right), \frac{d}{dp_x} = \frac{1}{2}\left(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}\right),$$

$$\frac{d}{dy} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}\right), \frac{d}{dp_y} = \frac{1}{2}\left(\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}\right). \quad (8)$$

Expressing $V(x,y)$ , $\psi(x,y)$ and $\lambda$ in terms of real and imaginary components as

$$V(x,y) = V_r(x_1,p_3,x_2,p_4) + iV_i(x_1,p_3,x_2,p_4),$$
$$\psi(x,y) = \psi_r(x_1,p_3,x_2,p_4) + i\,\psi_i(x_1,p_3,x_2,p_4),$$
$$\lambda = \lambda_r + i\lambda_i \quad , \quad (9)$$

where subscripts `r' and `i' denote the real and imaginary parts of the corresponding quantities and other subscripts to these quantities separated by comma will denote the partial derivatives of the quantity concerned. Thus, using (6) and (8) in (7), after separating real and imaginary parts, reduces to a pair of coupled partial differential equations as

$$-\frac{D}{4}\left[\psi_{r,x_1x_1} - \psi_{r,p_3p_3} + 2\psi_{i,x_1p_3} + \psi_{r,x_2x_2} - \psi_{r,p_4p_4} + 2\psi_{i,x_2p_4}\right] + \frac{v}{2}\left[\psi_{r,x_1} + \psi_{i,p_3} + \psi_{r,x_2} + \psi_{i,p_4}\right] + V_r\psi_r - V_i\psi_i = \lambda_r\psi_r - \lambda_i\psi_i\,, \quad (10)$$

$$-\frac{D}{4}\left[\psi_{i,x_1x_1} - \psi_{i,p_3p_3} - 2\psi_{r,x_1p_3} + \psi_{i,x_2x_2} - \psi_{i,p_4p_4} - 2\psi_{r,x_2p_4}\right] + \frac{v}{2}\left[\psi_{i,x_1} - \psi_{r,p_3} + \psi_{i,x_2} - \psi_{r,p_4}\right] + V_r\psi_i + V_i\psi_r = \lambda_i\psi_r + \lambda_r\psi_i\,. \quad (11)$$

The Cauchy-Riemann analyticity conditions for $\psi(x,y)$ may be written as

$$\psi_{r,x_1} = \psi_{i,p_3}\ ,\ \psi_{i,x_1} = -\psi_{r,p_3}\ ,\ \psi_{r,x_2} = \psi_{i,p_4}\ ,\ \psi_{i,x_2} = -\psi_{r,p_4}. \quad (12)$$

Hence, in view of (12), (10) and (11) reduces to

$$-D\left[\psi_{r,x_1x_1} + \psi_{r,x_2x_2} + v(\psi_{r,x_1} + \psi_{r,x_2})\right] + V_r\psi_r - V_i\psi_i = \lambda_r\psi_r - \lambda_i\psi_i, \quad (13)$$

$$-D\left[\psi_{i,x_1x_1} + \psi_{i,x_2x_2} + v(\psi_{i,x_1} + \psi_{i,x_2})\right] + V_r\psi_i + V_i\psi_r = \lambda_i\psi_r + \lambda_r\psi_i. \quad (14)$$

Here, we make an ansatz for the wave function $\psi(x,y)$ as

$$\psi(x,y) = \exp[g(x,y)] \quad (15)$$

Where, $g(x,y)$ is the function of the complex variables $x$ and $y$, expressed as

$$g(x_1,p_3,x_2,p_4) = g_r(x_1,p_3,x_2,p_4) + i\,g_i(x_1,p_3,x_2,p_4)\ , \quad (16)$$

After implying (16) in (15), the real and imaginary parts of $\psi(x,y)$ are written as

$$\psi_r(x_1,p_3,x_2,p_4) = e^{g_r}Cos\,g_i,$$
$$\psi_i(x_1,p_3,x_2,p_4) = e^{g_r}\,Sin\,g_i\,. \quad (17)$$

The analyticity conditions (12) for the function $g_r(x_1,p_3,x_2,p_4)$ and $g_i(x_1,p_3,x_2,p_4)$ becomes

$$g_{r,x_1} = g_{i,p_3}\ ,\ g_{i,x_1} = -g_{r,p_3}.$$
$$g_{r,x_2} = g_{i,p_4}\ ,\ g_{i,x_2} = -g_{r,p_4}. \quad (18)$$

Usng (17)- (18) in (13) - (14), we have

$$g_{r,x_1x_1} + g_{r,x_2x_2} + (g_{r,x_1})^2 + (g_{r,x_2})^2 - (g_{i,x_1})^2 - (g_{i,x_2})^2 - \frac{v}{D}(g_{r,x_1} + g_{r,x_2})\ ) + \frac{1}{D}(\lambda_r - V_r)\ = 0\ , \quad (19)$$

$$g_{i,x_1x_1} + g_{i,x_2x_2} + 2g_{r,x_1}g_{i,x_1} + 2g_{r,x_2}g_{i,x_2} - \frac{v}{D}(g_{i,x_1} + g_{i,x_2}) + \frac{1}{D}(\lambda_i - V_i) = 0, \quad (20)$$

It is to be noted that for given forms of $g_r$ and $g_i$ the rationalization of (19) and (20) provide the real and imaginary parts of the eigenvalue spectrum for a given system.

## III. Examples

In this section, we compute the eigenvalues and eigenfunctions for a shifted harmonic potential using the method described in section-II.

Shifted Harmonic Potential: Consider a shifted harmonic potential of the form

$$V(x,y) = ax + by + cx^2 + dy^2 + exy. \quad (21)$$

Where, a , b, c, d and e are complex constant

The real and imaginary parts of the potential, (21), using transformation (1), are written as

$$V_r = a_r\,x_1 - a_ip_3 + b_r\,x_2 - b_ip_4 + c_r(-{p_3}^2) - 2c_ix_1p_3 + d_r({x_2}^2 - {p_4}^2)\ - 2d_ix_2p_4 + e_r(x_1x_2 - p_3p_4) - e_i(x_1p_4 - x_2p_3), \quad (22)$$

$$V_i = a_ix_1 + a_rp_3 + b_ix_2 + b_rp_4 + c_i + 2c_rx_1p_3 + d_i({x_2}^2 - {p_4}^2)\ + 2d_rx_2p_4 + e_i(x_1x_2 - p_3p_4) + e_r(x_1p_4 - x_2p_3) \quad (23)$$

The ansatz for $g_r$ and $g_i$ for the underlying system, which conform the eq.(18), are considered as

$$g_r = \delta_1x_1 - \delta_2p_3 + \delta_3\,x_2 - \delta_4p_4\ + \frac{1}{2}\alpha_1({x_1}^2 - {p_3}^2) + \beta_1x_1p_3\ + \frac{1}{2}\alpha_2({x_2}^2 - {p_4}^2)\ + \beta_2x_2p_4 + \gamma_1(x_1x_2 - p_3p_4) - \gamma_2(x_1p_4 - x_2p_3), \quad (24)$$

$$g_i = \delta_2 x_1 + \delta_1 p_3 + \delta_4 x_2 + \delta_3 p_4 - \frac{1}{2}\beta_1({x_1}^2 - {p_3}^2) + \alpha_1 x_1 p_3 + \frac{1}{2}\beta_2({x_2}^2 - {p_4}^2) + \alpha_2 x_2 p_4 + \gamma_2(x_1 x_2 - p_3 p_4) - \gamma_1(x_1 p_4 - x_2 p_3). \quad (25)$$

Employing (22)-(25) in (19) - (20) and equating the coefficients of $x_1, p_3, x_2, p_4$ and their various products to zero, we obtain a set of non-repeating equations as

$$\lambda_r = D\left[\frac{v}{D}(\delta_1 + \delta_3) - \alpha_1 - \alpha_2 + {\delta_2}^2 - {\delta_1}^2 + {\delta_4}^2 - {\delta_3}^2\right], \quad (26)$$

$$\lambda_i = D\left[\frac{v}{D}(\delta_2 + \delta_4) + \beta_1 + \beta_2 - 2\delta_1\delta_2 - 2\delta_3\delta_4\right], \quad (27)$$

$${\gamma_1}^2 - {\gamma_2}^2 + {\alpha_1}^2 - {\beta_1}^2 = \frac{c_r}{D}, \quad (28)$$

$$\gamma_1\gamma_2 - \alpha_1\beta_1 = \frac{c_i}{2D}, \quad (29)$$

$${\gamma_1}^2 - {\gamma_2}^2 + {\alpha_2}^2 - {\beta_2}^2 = \frac{d_r}{D}, \quad (30)$$

$$\gamma_1\gamma_2 - \alpha_2\beta_2 = \frac{d_i}{2D}, \quad (31)$$

$$\gamma_1(\alpha_1 + \alpha_2) + \gamma_2(\beta_1 + \beta_2) = \frac{e_r}{2D}, \quad (32)$$

$$\gamma_2(\alpha_1 + \alpha_2) - \gamma_1(\beta_1 + \beta_2) = \frac{e_i}{2D}, \quad (33)$$

$$-\frac{v}{2D}(\alpha_1 + \gamma_1) + \alpha_1\delta_1 + \delta_2\beta_1 + \delta_3\gamma_1 - \delta_4\gamma_2 = \frac{a_r}{2D}, \quad (34)$$

$$-\frac{v}{2D}(\beta_1 - \gamma_2) + \alpha_1\delta_2 - \delta_1\beta_1 + \delta_3\gamma_2 + \delta_4\gamma_1 = \frac{a_i}{2D}, \quad (35)$$

$$-\frac{v}{2D}(\alpha_2 + \gamma_1) + \alpha_2\delta_3 + \delta_4\beta_2 + \delta_1\gamma_1 - \delta_2\gamma_2 = \frac{b_r}{2D}, \quad (36)$$

$$-\frac{v}{2D}(\beta_2 - \gamma_2) + \alpha_2\delta_4 - \delta_3\beta_2 + \delta_1\gamma_2 + \delta_2\gamma_1 = \frac{b_i}{2D}, \quad (37)$$

In order to find eigenvalues and the corresponding eigenfunction for the system, one should make some plausible choices among $\alpha$'s, $\beta's, \gamma's, \delta$'s while solving (28)-(37) in order to avoid any conflict between the general solutions and the well established results. Hence keeping such possibilities in mind, we choose $\gamma_1 = \gamma_2$ and $\gamma_1\gamma_2 = -\alpha_1\beta_1$ . Thus, for these choices, (28)-(31) immediately lead to

$$\alpha_1 = -c_+ , \beta_1 = -c_- , \alpha_2 = -d_+ ,$$

$$\beta_2 = -d_- , \gamma_1 = \gamma_2 = \frac{1}{2}\sqrt{\frac{c_i}{D}}, \quad (38)$$

where $c_+ = \sqrt{\frac{c_r + \sqrt{{c_r}^2 + {c_i}^2/4}}{2D}}$, $c_- = \sqrt{\frac{-c_r + \sqrt{{c_r}^2 + {c_i}^2/4}}{2D}}$

$$d_+ = \sqrt{\frac{d_r + \sqrt{{d_r}^2 + {d_i}^2/4}}{2D}}, \quad d_- = \sqrt{\frac{-d_r + \sqrt{{d_r}^2 + {d_i}^2/4}}{2D}} \quad (39)$$

Further, the (32) and (33) give two constraining relations on the choices of the potential coupling parameters a, b, c, etc. given by

$$\frac{1}{2}\sqrt{\frac{c_i}{D}}(c_- + d_- - c_+ - d_+) - \frac{e_r}{2D} = 0, \quad (40)$$

$$\frac{1}{2}\sqrt{\frac{c_i}{D}}(c_- + d_- + c_+ + d_+) + \frac{e_i}{2D} = 0. \quad (41)$$

Now in order to obtain the solutions for $\delta_i$ , i=1,2,3,4 we choose, for simplicity, $\delta_1 = -\delta_3$ , $\delta_2 = -\delta_4$ and utilizing the (34) and (35) we get

$$\delta_3 = -\delta_1 = \frac{\frac{v}{2D}\left[\frac{c_i}{2D} - \frac{1}{D}\sqrt{{c_r}^2 + {c_i}^2/4}\right] + \frac{1}{2D}\left[\frac{1}{2}(a_r + a_i)\sqrt{\frac{c_i}{D}} + a_r c_+ + a_i c_-\right]}{\frac{c_i}{2D} + \frac{1}{D}\sqrt{{c_r}^2 + {c_i}^2/4} + (c_- + c_+)\sqrt{\frac{c_i}{D}}}, \quad (42)$$

$$\delta_4 = -\delta_2 = \frac{-\frac{v}{2D}\sqrt{\frac{c_i}{D}}(c_- - c_+) + \frac{1}{2D}\left[a_i\left(\frac{1}{2}\sqrt{\frac{c_i}{D}} + c_+\right) - a_r\left(\frac{1}{2}\sqrt{\frac{c_i}{D}} + c_-\right)\right]}{\frac{1}{2D}\sqrt{{c_r}^2 + {c_i}^2/4} + \frac{c_i}{2D} + (c_- + c_+)\sqrt{\frac{c_i}{D}}}. \quad (43)$$

Thus, after substituting the ansatz parameters (38), (42) and (43) in (26), (27) the real and imaginary components of the energy eigenvalue are

$$\lambda_r = D\left[(c_+ + d_+) + 2({\delta_2}^2 - {\delta_1}^2)\right], \quad (44)$$

$$\lambda_i = D\left[(c_- + d_-) - 4\delta_1\delta_2\right], \quad (45)$$

and the corresponding eigenfunction is given by

$$\psi(x) = N\,exp\left[(\delta_1 + i\delta_2)(x - y) + \frac{1}{2}(c_+ - i\,c_-)x^2 + \frac{1}{2}(d_+ - i\,d_-)y^2 + \frac{1}{2}(1 + i)\sqrt{\frac{c_i}{D}}xy\right], \quad (46)$$

PT- Symmetric Case: Here, we consider the PT-symmetric version of the potential(21), which is obtained by setting $a_r = b_r = c_i = d_i = e_i = 0$ as

$$V(x, y) = a_i x + b_i y + c_r x^2 + d_r y^2 + e_r xy \quad (47)$$

The eigenvalues and the eigenfunction for this case can be obtained using the same ansatz for $g_r$ and $g_i$ used as in general case. As a result, the (26)-(37) reduce to some simpler forms as

$$\lambda_r = D\left[-\alpha_1 - \alpha_2 + {\delta_2}^2 + {\delta_4}^2\right], \quad (48)$$

$$\lambda_i = D\left[\frac{v}{D}(\delta_2 + \delta_4) + \beta_1 + \beta_2\right], \quad (49)$$

$${\gamma_1}^2 + {\alpha_1}^2 = \frac{c_r}{D}, \quad (50)$$

$${\gamma_1}^2 + {\alpha_2}^2 = \frac{d_r}{D}, \quad (51)$$

$$\gamma_1(\alpha_1 + \alpha_2) = \frac{e_r}{2D}, \quad (52)$$

$$\alpha_1\delta_2 + \delta_4\gamma_1 = \frac{a_i}{2D}, \quad (53)$$

$$\alpha_2\delta_4 + \delta_2\gamma_1 = \frac{b_i}{2D}, \quad (54)$$

In order to solve the (50)-(54), we choose $\gamma_1 = -\alpha_1$, then the ansatz parameters are obtained as

$$\gamma_1 = -\alpha_1 = \sqrt{\frac{c_r}{2D}}, \quad \alpha_2 = \sqrt{\frac{d_r}{D} - \frac{c_r}{2D}},$$

$$\delta_2 = -\delta_4 = \frac{a_i}{2}\sqrt{\frac{1}{2Dc_r}}. \quad (55)$$

Thus, after substituting the ansatz parameters (55) in (48) and (49), the real and imaginary components of energy eigenvalue are given by

$$\lambda_r = D\left[\sqrt{\frac{c_r}{2D}} + \sqrt{\frac{d_r}{D} - \frac{c_r}{2D}} - \frac{{a_i}^2}{4Dc_r}\right], \quad \lambda_i = 0, \quad (56)$$

Finally, the eigenfunction turns out to be

$$\psi(x) = N\,exp\left[i\frac{a_i}{2}(x - y) - \frac{1}{2}\sqrt{\frac{c_r}{2D}}x^2 - \frac{1}{2}\sqrt{\frac{d_r}{D} - \frac{c_r}{2D}}y^2 + \sqrt{\frac{c_r}{2D}}\,xy\right]. \quad (57)$$

## IV. Conclusions

In this paper, we have investigated the exact solutions of D-R equation to explore nontrivial applications of the extended complex phase space approach. For this purpose ansatz method is employed and potential coupling parameters are also taken complex besides the complexities generated by transformation (1). It is found that the imaginary component of the energy eigenvalues is non-zero only in the presence of complex coupling parameters and reduces to zero if the system is PT-symmetric. So present method suggest us another degree of freedom to obtain the real eigen spectra for non hermitian operator. Such restriction on the complex coupling constant will be in addition to what we already have in the form of constraining relation on the parameter of the potential which enable us to obtain the exact solution of the D-R equation for two dimensional complex Hamiltonian systems. This extension can be utilized to study some more realistic two dimensional systems. The extended phase space approach in two dimensions can be utilized to study more nontrivial potentials. However, for more involved complex systems, particularly in higher dimensions, studies may become a bit tedious due to the expansion of the algebra and difficulty in choosing the appropriate forms of the components $\psi$ $g_r$ and $g_i$ of the eigenfunction.

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