Application of Laplace Decomposition Method to Integro-Differential Equations

Jamshad Ahmad  
Department of Mathematics,  
Faculty of Sciences,  
University of Gujrat Pakistan  
jamshadahmadm@gmail.com

Sundas Rubab  
Department of Mathematics,  
Faculty of Sciences,  
University of Gujrat Pakistan

Iffat Siddique  
Govt. I. D. Jauwa College for Women, Gujrat Pakistan

Liaquat Tahir  
RACHNA College of Engineering & Technology, Gujranwala, Pakistan

Abstract – In this paper, we apply the proposed modified Laplace Adomian decomposition method (LADM) which is the coupling of Laplace transform and Adomian decomposition method by using He’s polynomial to solve the integro-differential equations. Several examples are tested and the results of the study are discussed. The obtained results explicitly reveal the complete reliability, efficiency, and accuracy of the proposed algorithm for solving the integro-differential equations and hence can be extended to other problems of diverse nature.

Keywords – Integral Equation, Integro-Differential Equation, Laplace Transforms, Adomian Decomposition Method.

I. INTRODUCTION

Nonlinear problems are widely used to describe complex physical phenomena in various fields of sciences, and engineering. They also cover the cases of the following types: surface waves in compressible fluids, hydro magnetic waves in cold plasma, acoustic waves in anharmonic crystal, etc. The wide applicability of these equations is the main reason why they have attracted so much attention from many mathematicians. However, such problems are usually very difficult to solve, either numerically or theoretically. Recently, both mathematicians and physicists have devoted considerable effort to the study of exact and numerical solutions of the nonlinear ordinary or partial differential equations corresponding to the nonlinear problems.

whose terms are determined by a recursive relation. Some fundamental works on various aspects of modifications of the Adomian’s decomposition method are given by Andrianov [2], Venkataraman [4,5] and Wazwaz [6]. Wazwaz [7] used the modified decomposition method and the traditional methods for solving nonlinear integral equations. A variety of powerful methods has been presented, such as the homotopy analysis method since Adomian firstly proposed the decomposition method [1] at the beginning of 1980s, the algorithm has been widely and effectively used for solving the analytic solutions of physically significant equations arranged from linear to nonlinear, from ordinary differential to partial differential, from integer to fractional integral equations. This method has since been termed the Adomian decomposition method (ADM) and has been the subject of many investigations [2,3]. This method generates a solution in the form of a series [8-10], homotopy perturbation method [11], the Exp-function method [12], variational iteration method [13] and the Adomian decomposition method [14].

Laplace Adomian’s Decomposition Method (LADM) was first introduced by Suheil A. Khuri [15,16], and has been successfully used to find the solution of differential equations [17-23]. This method has been applied successfully to find the exact solution of the Bratu and Duffing equation in [24,25]. The significant advantage of this method is its capability of combining the two powerful methods to obtain exact solution for nonlinear equation. Elgasery [26] applied the Laplace decomposition method for the solution of Falkner–Skan equation. Hussain and Khan in [27] the modified Laplace decomposition method has applied for solving some PDEs. Recently, the authors have used several methods for the numerical or the analytical solution of linear and nonlinear Fredholm and Volterra integral and integro-differential equations of the second kind [28]. The nonlinear Volterra integro-differential equations are given by

$$u^r(x) = f(x) + \int_{a}^{x} K(x,t)[Ru(t) + Nu(t)]dt$$

$$u^{(n)}(x) = a_n, 0 \leq k \leq (n-1), n \geq 0,$$

and the nonlinear Fredholm integro-differential equations are given by

$$u^n(x) = f(x) + \int_{a}^{b} K(x,t)[Ru(t) + Nu(t)]dt,$$

$$u^{(k)}(x) = a_k, 0 \leq k \leq (n-1), n \geq 0,$$

Where $u^n(x)$ is the nth derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the integral equation, $f(x)$ is an analytic function, $R(u)$ and $N(u)$ are linear and nonlinear functions of $u$, respectively. Our aim in this paper is to obtain the analytical solutions by using the Laplace decomposition method.

II. ANALYSIS OF LAPLACE DECOMPOSITION METHOD (LDM)

The purpose of this section is to discuss the use of modified Laplace decomposition algorithm for the integro-differential equations. We consider the general form of second order nonlinear partial differential equations with initial conditions in the form

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t),$$

$$u(x,0) = f(x), u_t(x,0) = g(x).$$

Copyright © 2014 IJISM, All right reserved
Where \( L \) is the second order differential operator, \( R \) is the remaining linear operator, \( N \) represents a general nonlinear differential operator and \( h(x, t) \) is the source term. Applying Laplace transform to both sides of Eq. (3) we have

\[
L[Lu(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[h(x,t)],
\]

\[
s^2L[Lu(x,t)] - sL[h(x,t)],
\]

\[
\frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2} L[Ru(x,t)] + L[Nu(x,t)] = L[h(x,t)],
\]

(4)

Assume the solution of Eq. (4) to be in the form

\[
u(x, t) = \sum_{j=0}^{\infty} p^j H_j,
\]

where \( H_n \)’s polynomial given by

\[
H_n(u_0, \ldots, u_n) = \frac{\partial^n}{\partial x^n} (N(\sum_{i=0}^{\infty} p^i u_i))_{p=0}, n = 0,1,2,\ldots
\]

Using (4), (5) and (6) we get

\[
\sum_{i=0}^{\infty} L(p^i u_i) = \frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2} \sum_{j=0}^{\infty} L(R(p^j u_j))
\]

(7)

Comparing both sides of (7) we have

\[
p^0; L[u_0(x,t)] = k_0(x, s),
\]

(8)

\[
p^1; L[u_1(x,t)] = k_1(x,s) - \frac{1}{s} L[R_0 u(x,t)] - \frac{1}{s} L[H_0(x,t)],
\]

(9)

\[
\vdots
\]

\[
p^n; L[u_n(x,t)] = -\frac{1}{s} L[R_{n-1} u(x,t)] - \frac{1}{s} L[H_n(x,t)], n \geq 1.
\]

(10)

Applying the inverse Laplace transform to Eq. (8)–(10) gives our required recursive relation as follows

\[
p^0; u_0(x,t) = k_0(x,t),
\]

(11)

\[
p^1; u_1(x,t) = k_1(x,t) - L^{-1}\left[\frac{1}{s} R_0 u(x,t)\right] - L^{-1}\left[\frac{1}{s} H_0(x,t)\right]
\]

(12)

\[
\vdots
\]

\[
p^n; u_n(x,t) = L^{-1}\left[\frac{1}{s} R_{n-1} u(x,t)\right] - L^{-1}\left[\frac{1}{s} H_n(x,t)\right]
\]

(13)

The solution through the modified Adomian decomposition method highly depends upon the choice of \( k_0(x, t) \) and \( k_1(x, t) \), where \( k_0(x, t) \) and \( k_1(x, t) \) represent the terms arising from the source term and prescribed initial conditions.

### III. APPLICATION OF THE LAPLACE DECOMPOSITION METHOD (LDM)

In this section we give four examples to illustrate efficiency and applicability of LDM method for the integro-differential equations.

**Example 3.1:** Consider the nonlinear integro-differential equation [29]

\[
u'(x) = -1 + \int_0^x u^2(t)dt, u(0) = 0,
\]

(14)

Applying the Laplace transforms and by using the initial condition, we have

\[
su(s) = -\frac{1}{s^2} + \frac{1}{s} \int_0^x u^2(t)dt,
\]

(15)

Applying the inverse Laplace transform to Eq. (15), we get

\[
u(x) = -x + \frac{1}{s} \int_0^x u^2(t)dt,
\]

(16)

We decompose the solution as an infinite sum given below

\[
u(x) = \sum_{i=0}^{\infty} p^i u_i.
\]

(17)

Using Eq. (17) into Eq. (16) we get

\[
\sum_{i=0}^{\infty} p^i u_i = -x + \frac{1}{s} \left[\frac{1}{s} \left(\sum_{n=0}^{\infty} H_n(t)dt\right)\right],
\]

(18)

The recursive relation is given below

\[
p^0; u_0(x) = -x,
\]

(19)

\[
p^1; u_1(x) = \frac{x^2}{12},
\]

\[
p^2; u_2(x) = -\frac{x^5}{288},
\]

\[
\vdots
\]

The solution is read as \( p \to 1 \)

\[
u(x) = -x + \frac{x^2}{12} - \frac{x^5}{288} + \ldots
\]

(20)

**Example 3.2:** Consider the second–order nonlinear integro differential equation [30]

\[
u''(x) = \sinh(x) + x - \int_0^x (\cosh^2(t) - u^2(t))dt,
\]

(21)

With subject to initial conditions

\[
u(0) = 0, u'(0) = 1.
\]

Applying the Laplace transforms and by using the initial conditions we obtain

\[
s^2u(s) - 1 = \frac{1}{s^3} + \frac{1}{s} \left[\frac{1}{s} \left(\int_0^x (\cosh^2(t) - u^2(t))dt\right)\right],
\]

(22)
Consider the second-order linear integro-differential equation [31]

\[ u''(x) + xu'(x) - xu(x) = e^x - 2 \sin x + \int_{-1}^{x} \sin x e^{-3t}u(t)dt, \]  

Subject to initial condition
\[ u(0) = 1, u'(0) = 1. \]

Applying the Laplace transforms and by using the initial conditions, we have

\[ (s^2 - 1)u(s) - (s - 1)\frac{du}{dt}(0) - s - 1 = \frac{1}{s^2} - \frac{1}{s} + L \left[ \int_{-1}^{x} \sin x e^{-3t}u(t)dt \right]. \]

Applying the inverse Laplace transform, we get
\[ u(x) = e^x + L^{-1} \left[ \frac{1}{s^2} \frac{du}{dt}(s) + \frac{1}{s(1+(s-1)^2)} - \frac{2}{(s^2+1)^2} - \frac{1}{(s^2-1)}L \left[ \int_{-1}^{x} \sin x e^{-3t}u(t)dt \right] \right]. \]

Consequently, we have
\[ p^0; u_0(x) = e^x, \]
\[ p^1; u_1(x) = L^{-1} \left[ \frac{1}{s^2} \frac{du}{dt}(s) + \frac{1}{s(1+(s-1)^2)} - \frac{2}{(s^2+1)^2} - \frac{1}{(s^2-1)}L \left[ \int_{-1}^{x} \sin x e^{-3t}u(t)dt \right] \right], \]
\[ \vdots \]
\[ p^n; u_n(x) = 0, n \geq 1. \]

The solution in the series form is given by
\[ u(x, t) = \sum_{i=0}^{\infty} p_i u = u_0 + pu_1 + p^2 u_2 + \ldots, \]
\[ u(x, t) = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \ldots. \]

The closed form solution is
\[ u(x) = e^x. \]  

**Example 3.4:** Consider the second-order linear integro-differential equation [31]

\[ u''(x) + xu'(x) - xu(x) = e^x - 2 \sin x + \int_{-1}^{x} \sin x e^{-3t}u(t)dt, \]  

Subject to initial condition
\[ u(0) = 1, u'(0) = 1. \]

Applying the Laplace transforms and by using the initial conditions, we have

\[ (s^2 - 1)u(s) - (s - 1)\frac{du}{dt}(0) - s - 1 = \frac{1}{s^2} - \frac{1}{s} + L \left[ \int_{-1}^{x} \sin x e^{-3t}u(t)dt \right]. \]

Applying the inverse Laplace transform, we get
\[ u(x) = e^x + L^{-1} \left[ \frac{1}{s^2} \frac{du}{dt}(s) + \frac{1}{s(1+(s-1)^2)} - \frac{2}{(s^2+1)^2} - \frac{1}{(s^2-1)}L \left[ \int_{-1}^{x} \sin x e^{-3t}u(t)dt \right] \right]. \]

Consequently, we have
\[ p^0; u_0(x) = e^x, \]
\[ p^1; u_1(x) = L^{-1} \left[ \frac{1}{s^2} \frac{du}{dt}(s) + \frac{1}{s(1+(s-1)^2)} - \frac{2}{(s^2+1)^2} - \frac{1}{(s^2-1)}L \left[ \int_{-1}^{x} \sin x e^{-3t}u(t)dt \right] \right], \]
\[ \vdots \]
\[ p^n; u_n(x) = 0, n \geq 1. \]

The solution in the series form is given by
\[ u(x, t) = \sum_{i=0}^{\infty} p_i u = u_0 + pu_1 + p^2 u_2 + \ldots, \]
\[ u(x, t) = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \ldots. \]

The closed form solution is
\[ u(x) = e^x. \]  

**IV. CONCLUSION**

In this paper, we carefully applied a reliable modification of Laplace decomposition method using He’s polynomials for solving integro-differential equations. The main advantage of this method is the fact that it gives the analytical solution. The method overcomes the difficulties arising in calculating the Adomian polynomials. The efficiency of the method was tested on some numerical examples, and the results show that the method is easier than many other numerical techniques. We note that solution which obtained solutions by the LDM are the same [11, 29]. It is also observed that LDM is a promising tool for linear and nonlinear integro-differential equations.
REFERENCES


