International Journal of Innovation in Science and Mathematics
Volume 2, Issue 5, ISSN (Online): 2347–9051

A New Concept for the Solution of Nonlinear Differential Equations

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***Abstract* – Present study introduces a concept for the solution of a class of nonlinear differential equations. An equation of this class is thought as a combination of some linear differential equations that rules the formation of complicated solution of the nonlinear equation from simple solutions of those linear equations. Exact or approximate analytic solutions of these nonlinear differential equations are obtained here by a method due to the new concept called the *method of matrix inner product*. Considered class of nonlinear differential equations encompasses both ordinary and partial differential equations with constant and variable coefficients. Apart from the obtained exact solutions, the approximate solutions of the nonlinear differential equations in general are found to be good at least in predicting the trend of the known solutions.**

***Keywords* – Nonlinear Differential Equation, Nonlinearity Constant, Matrix Inner Product, Exact Solution, Approximate Solution.**

I. Introduction

Mathematical formulation of almost all the physical problems gives rise to differential equations which are often nonlinear. But very little about the general character of nonlinear equations is known despite the general theory and methods of linear equations are highly developed. Study of nonlinear differential equations is confined to a variety of special cases, e.g. [1-3] and their general solutions are rarely obtainable, though particular solutions can be calculated by standard numerical method.

It is a simple fact that the solution of a differential equation remains manifested in the equation as a dependent variable and its derivatives. Because the general solution upon its substitution satisfies the differential equation completely. A class of nonlinear differential equation is thought to be a combination of linear differential equations, consisting of those dependent variable and derivatives, leading to a new concept that is *a nonlinear differential equation is a combination of non-identical linear differential equations where these linear differential equations and their solutions as well may combine as inner product of matrices to form the original nonlinear differential equation and its solution.* The justification of this new concept is sought through an algebraic equation in the next section.

The primary feature of nonlinear differential equations is that they do not admit superposition principle which is not in contradiction to the combination considered in the new concept. The class of nonlinear differential equations under consideration has the characteristic that the equation can be split into two or more dissimilar linear differential equations. The higher order derivatives with the associated coefficients taken from the terms of the nonlinear differential equation constitute the first linear equation and the remainder constitutes the second linear equation such that the linear equations retain all the dependent variable and derivatives of the nonlinear equation. The original nonlinear differential equation can be obtained by combining those linear differential equations as inner product of matrices and its solution can be obtained by combining the solutions of those linear differential equations in the same manner. This way of solving the nonlinear differential equations may be called the *method of matrix inner product* (MIP). Present study on the solution of nonlinear differential equations is mainly concerned with solving the ordinary differential equations (ODEs). Some discussion in Sec. 4.8 brings out the difficulty and possibility of solving the nonlinear partial differential equations (PDEs) by the proposed method.

II. Algebraic Basis of the Concept

Justification for the new concept to the solution of nonlinear differential equations is sought through an algebraic equation

$$x^2 + 2y^2 = 6 \tag{1}$$

where $(2,1)$ is a possible set of values satisfying this equation. Hence Eq. (1) can be written as

$$\begin{bmatrix} x & 2y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}^T = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}^T \tag{2}$$

where T on a row matrix denotes its transpose. Equation (1) can be split into two dissimilar linear equations with the help of Eq. (2) as

$$x + 2y = 4\,, \qquad x + y = 3 \tag{3}$$

and one can obtain solutions of $x=2$ and $y=1$ from Eq. (3). Similarly, the algebraic equation

$$x^2 + y^2 = 5 \tag{4}$$

can be split into two identical linear equations as

$$x + y = 3\,, \qquad x + y = 3 \tag{5}$$

where Eq. (5) cannot provide solutions of x and y like Eq. (3). Hence these examples can be treated as the algebraic basis of the new concept for the solution of nonlinear differential equations.

III. Solution Procedure

The class of nonlinear differential equations considered here can be split into two or more linear differential equations, and their solution procedure is as follows.

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Volume 2, Issue 5, ISSN (Online): 2347–9051

3.1 A set of linear differential equations are made out of the nonlinear differential equation as described in previous section and their solutions are obtained by the known methods, and further those are combined by the inner product of matrices to form the original nonlinear differential equation as well as its complete solution. The terms of the solution in row matrix are ordered as they appear during integration for the homogeneous solution and the terms of the particular solution are in the same order of inhomogeneous terms of the linear differential equation in the row matrix as in Sec. 4.1.

3.2 A basic feature of the solution of nonlinear equation is that it does not get linear shift with the change of boundary conditions. If the constants of integration of the solutions obtained in Sec. 3.1 appear in a way lacking that feature then an additional arbitrary constant, may be called the *constant of nonlinearity*, is imposed such that it becomes a function of the constants of integration. In addition, the boundary conditions may guide the formation of the solution of the differential equation.

3.3 If substitution of the solution, obtained in Sec. 3.2, into the original nonlinear differential equation reduces it to an algebraic one in terms of constants of integration and constant of nonlinearity, then the solution is an exact analytic solution where the constant of nonlinearity can be expressed in terms of constants of integration.

3.4 If substitution of the solution does not reduce the original nonlinear equation, as in Sec. 3.3, to an algebraic one in terms of constants of integration and constant of nonlinearity, then with such substitution the original nonlinear equation may be transformed into a polynomial where each coefficient of the polynomial is an algebraic expression in terms of constants of integration and constant of nonlinearity. In order to satisfy the original differential equation by the obtained solution for any value of the independent variable, each of the coefficients including the constant term of the polynomial is assumed to be zero. The constant term or the suitable one among the coefficients may be used to express the constant of nonlinearity in terms of constants of integration. Thus the obtained solution is an approximate analytic solution.

IV. Application of the New Method

Solutions of some nonlinear differential equations of the considered class are obtained by the application of MIP method as examples and presented in this section to illustrate their solution procedure as well. Here a prime on y denotes differentiation once with respect to x or t and so forth for ODEs.

4.1 Nonlinear Differential Equation $y''^2 - y' = 0$

with associated boundary conditions $y(0) = 2$ and $y'(0) = 1$.

Its first and second linear equations, respectively, are

$$y'' - y' = 0, \qquad y'' + 1 = 0.$$

The detailed description of the solutions of the two linear differential equations is as follows. The first linear equation has no inhomogeneous term and integrating it twice

$$\ell n(b_1 + y) = x + b_2$$

which can be written as

$$y = -b_1 + e^{x+b_2} = a_1 + a_2 e^x$$

where a_1 and a_2 are the constants of integration. The second linear equation has one inhomogeneous term and integrating its homogeneous part twice

$$y = a_3 x + a_4$$

where a_3 and a_4 are the constants of integration, and the particular solution is

$$y = -x^2 / 2.$$

This particular solution takes its position in the complete solution in the order the inhomogeneous term appears in the second linear equation. So the complete solution of the second linear differential equation is

$$y = a_3 x + a_4 - x^2 / 2.$$

Hence, solutions of the first and second linear differential equations, respectively, are

$$y = a_1 + a_2 e^x, \qquad y = a_3 x + a_4 - x^2 / 2$$

where a_1, a_2, a_3 and a_4 are the constants of integration. The linear differential equations can be combined to form the original differential equation as

$$\begin{bmatrix} y'' & -y' \end{bmatrix} \begin{bmatrix} y'' & 1 \end{bmatrix}^T = 0$$

and solutions of the linear differential equations can be combined in the same manner to obtain the complete solution as

$$y = \begin{bmatrix} a_1 & a_2 e^x & 0 \end{bmatrix} \begin{bmatrix} a_3 x & a_4 - x^2/2 \end{bmatrix}^T$$

$$y = c_1 x + c_2 e^x$$

which may be written further as

$$y = c_1 x + c_2 e^{cx}$$

where c_1 and c_2 are the constants of integration and c is the constant of nonlinearity. Substitution of this solution into the original differential equation and through expansion results in a polynomial. Equating its constant term to zero gives $4c^4 - c_1 - 2c = 0$ for $c_2 = 2$ due to $y(0) = 2$ and this expression with other boundary condition yields $c_1 = 1 - \sqrt{2}$ and $c = 1/\sqrt{2}$ and thus the approximate solution becomes

$$y = c_1 x + 2e^{cx}.$$

4.2 Nonlinear Differential Equation $yy'' - y'^2 = 0$

Its first and second linear equations are

$$y'' - y' = 0, \qquad y + y' = 0$$

and their solutions are

$$y = a_1 + a_2 e^x, \qquad y = a_3 e^{-x}$$

where a_1, a_2 and a_3 are the constants of integration. The linear differential equations can be combined to form the original differential equation as

$$\begin{bmatrix} y'' & -y' \end{bmatrix} \begin{bmatrix} y & y' \end{bmatrix}^T = 0$$

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Volume 2, Issue 5, ISSN (Online): 2347–9051

and solutions of the linear differential equations can be combined in the same manner to obtain the complete solution as

$$y=\begin{bmatrix}a_1 & a_2 e^{x}\end{bmatrix}\begin{bmatrix}a_3 e^{-x} & 0\end{bmatrix}^T = c_1 e^{-x}$$

which lacks one more constant of integration. The solution may be written as

$$y = c_1 e^{-cx} + c_2$$

to ensure two constants of integration for a second order differential equation where c_1 and c_2 are the constants of integration and c is the constant of nonlinearity. Substitution of this solution into the original differential equation results in $c_2 = 0$ and the solution becomes an exact solution as

$$y = c_1 e^{-cx}$$

where c is acting both as constant of nonlinearity and constant of integration.

4.3 Nonlinear Differential Equation

$$y'' + \varepsilon\left(\tfrac{1}{3} y'^3 - y'\right) + y = 0$$

with associated boundary conditions $y(0) = a$ and $y'(0) = 0$.

This is the well known Rayleigh equation [4] for small value of ε. Its first, second and third linear equations are

$$y'' + \tfrac{1}{3}\varepsilon y' - \varepsilon y' + y = 0,$$

$$1 + y' + 1 + 1 = 0,$$

$$1 + y' + 1 + 1 = 0.$$

It is observed that solutions of these linear equations upon combining as matrix inner product do not yield solution of the nonlinear differential equation. This is in accordance with the statement of the new concept that is the linear equations required to be non-identical. Hence, those linear equations are re-written as the first and second linear equations as

$$y'' - \tfrac{2}{3}\varepsilon y' + y = 0, \qquad y' = \pm\sqrt{3}i.$$

The solutions of the first and second linear differential equations are

$$y = e^{\alpha t}\left(a_1 Cos\beta t + a_2 Sin\beta t\right), \; y = a_3 \pm \sqrt{3}it$$

where a_1, a_2 and a_3 are the constants of integration, $\alpha = \varepsilon / 3$ and $\beta = \sqrt{1-\varepsilon^2/9}$. The complete solution of the original differential equation by MIP method is

$$y = c_1 e^{\alpha t} Cos\left(\beta t + c_2\right)$$

where the particular solution $\pm\sqrt{3}it$ of the second linear differential equation can be placed on either side of its homogenous solution a_3 but needs to satisfy the boundary condition, imaginary part is omitted, c_1 and c_2 are the constants of integration and position of c_2 is taken arbitrarily to act also as the constant of nonlinearity. The boundary conditions provide $c_1 = a / Cos\, c_2$ and $c_2 = tan^{-1}(\alpha/\beta)$. Thus the approximate solution is

$$y = c_1 e^{\alpha t} Cos\left(\beta t + c_2\right).$$

4.4 Nonlinear Differential Equation

$$y''' + yy'' + y'^2 = 0$$

with associated boundary conditions $y(0) = y'(\infty) = y''(0) = 0$.

This is the equation for two-dimensional plane jet and its first and second linear equations are

$$y''' + y'' + y' = 0, \qquad 1 + y + y' = 0.$$

The solutions of the first and second linear differential equations are

$$y = a_1 + a_2 e^{-x/2} Cos\alpha x + a_3 e^{-x/2} Sin\alpha x,$$

$$y = -1 + a_4 e^{-x}$$

where a_1, a_2, a_3 and a_4 are the constants of integration, and $\alpha = \sqrt{3}/2$. The complete solution of the original differential equation by MIP method is

$$y = c_1 + c_2 e^{-x} Cos\left(\alpha x + c_3\right)$$

where c_1, c_2 and c_3 are the constants of integration and c_3 also acts as the constant of nonlinearity. Substitution of this solution into the original differential equation along with the given boundary conditions provide $c_1 = 2$, $c_2 = -c_1 / Cos\, c_3$ and $c_3 = tan^{-1}\left[\left(\alpha^2 - 1\right)/2\alpha\right]$. Thus the approximate solution becomes as

$$y = c_1 + c_2 e^{-x} Cos\left(\alpha x + c_3\right).$$

4.5 Nonlinear Differential Equation

$$y''' + yy'' + \beta(1 - y'^2) = 0$$

with associated boundary conditions $y(0) = y'(0) = 0$ and $y'(\infty) = 1$.

This is the well known Falkner-Skan equation [5] and its first and second linear equations are

$$y''' + y'' - \beta y' + \beta = 0, \qquad 1 + y + y' + 1 = 0.$$

The solutions of the first and second linear differential equations are

$$y = a_1 + a_2 e^{-\alpha x} + a_3 x, \qquad y = -2 + a_4 e^{-x}$$

where a_1, a_2, a_3 and a_4 are the constants of integration and $\alpha = \left(1 \mp \sqrt{1+4\beta}\right)/2$. The complete solution of the original differential equation may be

$$y = \begin{bmatrix}a_1 & a_2 e^{-\alpha x} & a_3 x\end{bmatrix}\begin{bmatrix}-1 & a_4 e^{-x} & -1\end{bmatrix}^T$$

$$y = c_1 + c_2 e^{-cx} + c_3 x$$

where c_1, c_2 and c_3 are the constants of integration and c is the constant of nonlinearity. Substitution of this solution into the original differential equation and through expansion results in a polynomial, and upon equating its constant term to zero along with the given boundary conditions provide $c_1 = -1/c$, $c_2 = 1/c$, $c_3 = 1$ and $c = \pm\sqrt{\beta}$. Thus the approximate bounded solution becomes

$$y = \sqrt{\beta}^{-1}\left(e^{-\sqrt{\beta}x} - 1\right) + x .$$

4.6 Nonlinear Differential Equation

$$x^2 y'' + y'^2 - 2xy' = 0$$

with the boundary conditions $y(2) = 5$ and $y'(2) = 2$. Its first and second linear equations are

$$x^2 y'' + xy' - 2xy' = 0, \qquad 1 + x^{-1} y' + 1 = 0 .$$

The original differential equation is a nonlinear ODE with variable coefficients. The beauty in its solution procedure is that the first linear equation can be formed as equidimensional one whose general solution is easily obtainable. The above two linear equations can be written further as

$$x^2 y'' - xy' = 0, \qquad y' + 2x = 0 .$$

The solutions of the first and second linear differential equations are

$$y = a_1 + a_2 x^2 , \qquad y = -x^2 + a_3$$

where a_1, a_2 and a_3 are the constants of integration. The complete solution of the original differential equation by MIP method is

$$y = a_4 x^2 + a_5 x^2 = c_1 x^2$$

which lacks one more constant of integration. The solution may be written as

$$y = c_1 (x + c)^2 + c_2$$

to ensure two constants of integration c_1 and c_2 for the second order differential equation where c is the constant of nonlinearity. Substitution of this solution into the original differential equation and equating the coefficients of x to zero provides $8c_1^2 c - 4c_1 c = 0$, resulting in $c_1 = 1/2$. The boundary conditions lead to $c = 0$ and $c_2 = 3$, thus the exact solution satisfying the original differential equation completely is

$$y = \tfrac{1}{2} x^2 + 3 .$$

4.7 Nonlinear Differential Equation

$$yy'y'' - y'^3 - y''^2 = 0$$

This equation can be split into the first, second and third linear equations as

$$y'' - y' + y'' = 0,$$

$$y' + y' - y'' = 0,$$

$$y + y' + 1 = 0 .$$

The solutions of the first, second and third linear differential equations are

$$y = a_1 e^{x/2} + a_2 ,$$

$$y = a_3 e^{2x} + a_4 ,$$

$$y = a_5 e^{-x} - 1$$

where a_1, a_2, a_3, a_4 and a_5 are the constants of integration. The linear differential equations can be combined to form the original differential equation as

$$\begin{bmatrix} y'' & -y' & y'' \end{bmatrix}\left[\begin{pmatrix} y' & y' & -y'' \end{pmatrix}\begin{pmatrix} y & y' & 1 \end{pmatrix}^T\right]^T = 0$$

and their solutions can be combined in the same manner to obtain the complete solution as

$$y = \begin{bmatrix} a_1 e^{x/2} & a_2 \end{bmatrix}\left[\begin{pmatrix} a_3 e^{2x} & a_4 \end{pmatrix}\begin{pmatrix} a_5 e^{-x} - 1 \end{pmatrix}^T\right]^T .$$

$$y = c_1 e^{3x/2} + c_2 .$$

The solution may be written as

$$y = c_1 e^{cx} + c_2$$

where c_1 and c_2 are the constants of integration and c is the constant of nonlinearity. Substitution of this solution into the original differential equation results in $c = c_2$ and the solution becomes an exact solution as

$$y = c_1 e^{c_2 x} + c_2 .$$

4.8 Nonlinear Partial Differential Equation

$$u_t + uu_x - u_{xx} = 0$$

This is the well known Burgers' equation [6] where u_t and u_x are the first order partial derivatives of u with respect to temporal and spatial co-ordinates t and x, respectively, and similar is u_{xx}. The initial and boundary conditions for this equation over the domain $0 \le x \le 1$ and $t > 0$ are

$$u(x,0) = \phi(x), \qquad u(0,t) = u(1,t) = 0 .$$

The first and second linear equations by splitting the original equation are

$$u_{xx} - u_x - 1 = 0 , \qquad 1 + u + u_t = 0$$

and their solutions are

$$u = a_1(t) + a_2(t)e^x - 1 , \qquad u = -1 + a_3(x)e^{-t} .$$

Complete solution of the original PDE by inner product of matrices becomes

$$u(x,t) = c_1(t) + c_2(t)\, c_3(x) e^{x-t}$$

which is a trivial solution. It seems that a solution of nonlinear PDE remains so as long as x and t are not related through the linear equation or equations.

The nonlinear PDE may be split into linear PDEs such that x and t are related through the linear equation or equations. Such linear equations from splitting the Burgers' equation are

$$u_{xx} - u - 1 = 0 , \qquad 1 + u_x + u_t = 0$$

and their solutions are

$$u = a_1(t)e^{-x} + a_2(t)e^x - 1 ,$$

$$u = a_4 - x + a_3 e^{\lambda(x-t)}$$

where a_1 and a_2 are functions of time, and a_3 and a_4 are constants. Complete solution through inner product of matrices is

$$u(x,t) = c_1(t)e^{-x} + c_2(t)xe^x + c_3 e^{\lambda(x-t)} .$$

In the solution of second linear equation, λ is a positive constant for the solution to be bounded in time and this constant may play the role of constant of nonlinearity as it does in nonlinear ODEs. But λ being a constant cannot be a function of x and t, and hence c_1, c_2 and c_3. In the complete solution, x and t are related through second

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Volume 2, Issue 5, ISSN (Online): 2347–9051

linear equation where c_1, c_2 and c_3 are evaluated from the given initial and boundary conditions with $\lambda=1$ as

$$c_1(t)=-e^{-t}\phi(x)/D(x),$$

$$c_2(t)=\left(e^{-2}-1\right)e^{-t}\phi(x)/D(x),$$

$$c_3=\phi(x)/D(x)$$

where $D(x)=e^{x}-e^{-x}+\left(e^{-2}-1\right)xe^{x}$. Now it appears that c_3 may be constant approximately depending on $\phi(x)/D(x)$. Hence the complete solution is

$$u(x,t)=c_1(t)e^{-x}+c_2(t)xe^{x}+c_3e^{x-t}.$$

V. Discussion

A large class of nonlinear ODEs in Secs. 4.1-4.7, both with constant and variable coefficients are solved by the application of the new method. Solutions of the problems in Secs. 4.2, 4.6 and 4.7 obtained by MIP method are exact analytic as the solutions satisfy the original differential equations completely without any restriction. The approximate solutions of the problems in Secs. 4.1, 4.3-4.5 are compared with the available solutions due to other methods in this section in Figs. 1-4 where y' denotes fluid velocity. The nonlinear PDE in Sec. 4.8 is solved by MIP method and the obtained approximate solution is compared with its numerical solution in Fig. 5.

Figure 1 shows the comparison of the exact analytic solution

$$y=(x+1/4)/(x+1/2)$$

and approximate analytic solution by MIP method of the problem in Sec. 4.1. The problem in Sec. 4.3 describes a nonlinear oscillator by Rayleigh equation and its approximate solution due to Kryloff and Bogoliuboff (K-B) method [2]

$$y=2Cost/\sqrt{\left[1-\left(1-4/a^{2}\right)e^{-\varepsilon t}\right]}$$

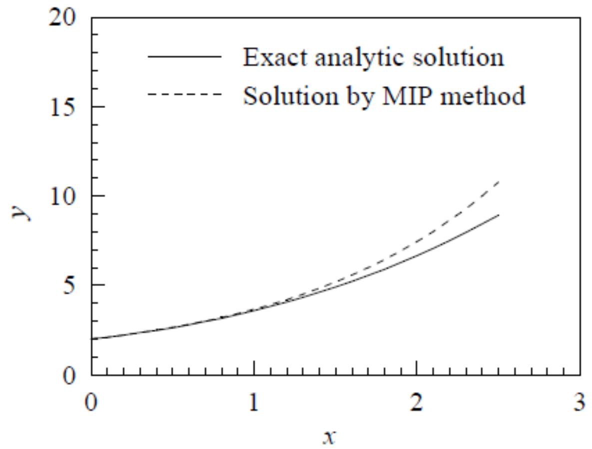

Fig.1. Solution $y(x)$ as a function of x

and approximate solution due to MIP method for $a=1$ and $\varepsilon=0.1$ are plotted in Fig. 2 for comparison, and found to be in good agreement. This good agreement is due to that one constant of integration of the present approximate solution which is acting as the constant of nonlinearity as well is evaluated using the given boundary condition rather than substituting the present approximate solution into the original equation. Problem in Sec. 4.4 describes a plane laminar free jet [7]. Its numerical solution obtained by the Fourth Order Runge Kutta method and approximate analytic solution[1] obtained by MIP method, which is

$$y'=-c_2e^{-x}\left[Cos(\alpha x+c)+\alpha Sin(\alpha x+c)\right],$$

are compared in Fig. 3. If constant of integration c, also acting as constant of nonlinearity, is placed in the form

$$y'=-c_2e^{-cx}\left(Cos\,\alpha x+\alpha Sin\,\alpha x\right)$$

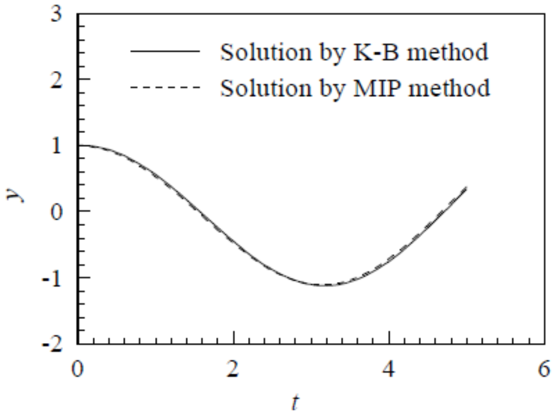

Fig.2. Solution y(t) as a function of t.

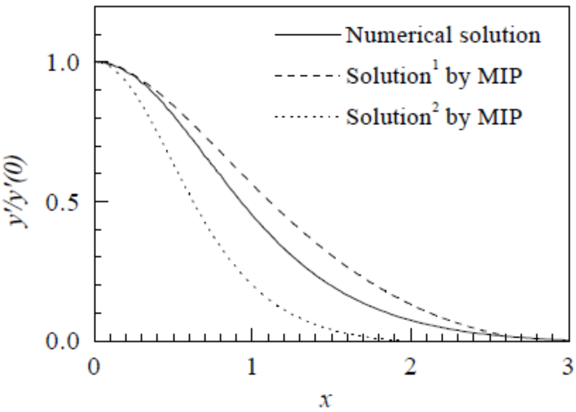

Fig.3. Solution y'(x) as a function of x.

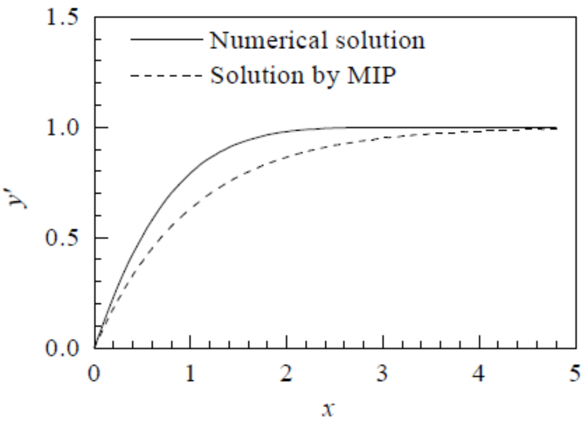

Fig.4. Solution y'(x) as a function of x.

then the agreement between the approximate analytic solution[2] and the numerical solution becomes poor. The problem in Sec. 4.5 describes Falkner-Skan type flow. Its analytic solution obtained by MIP method is compared with the numerical solution for $\beta=1$ in Fig. 4. The problem in Sec. 4.8 is the one dimensional Burgers' equation. The equation is solved here numerically using Crank-Nicolson Scheme [8] and Tridiagonal Matrix Algorithm [9] for the given boundary conditions. Its numerical solution and approximate analytic solution due to MIP method are obtained for $\phi(x)=x(1-x)$ and are compared in Fig. 5.

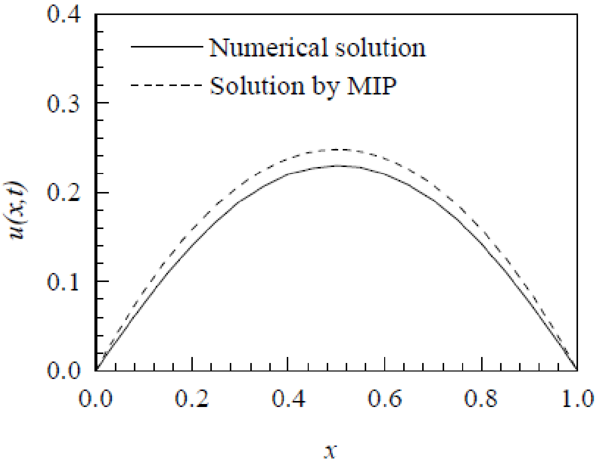

Fig.5. Solution of Burgers' equation at $t = 0.01$.

VI. Conclusion

Present study considers a class of nonlinear differential equations that can be split up into two or more linear differential equations. Exact or approximate analytic solutions of this class of differential equations are obtained by MIP method. Comparison of the obtained approximate solutions with the available solutions due to other methods which are either exact or approximate analytic or numerical shows the effectiveness of MIP method for solving nonlinear ODEs as well as PDEs.

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