Graceful Labeling for Tensor Product of Two Path of Odd Lengths and Some Grid Related Graphs

V. J. Kaneria
Department of Mathematics
Saurashtra University, Rajkot-360005
Email: kaneria_vinodray_j@yahoo.co.in

H. M. Makadia
Government Engineering College,
Rajkot-360005
Email: makadia.hardik@yahoo.com

Abstract – In this paper we have investigated gracefulfulness of tensor product of graphs and some grid related families. We obtained graceful labeling for tensor product of $P_m$ and $P_n$, when $m$ and $n$ both are odd. We also obtained graceful labeling for $t$–super subdivision of the grid graph $P_m \times P_n$ and path union of cycle $C_p$ ($p \equiv 0 \pmod{4}$) with grid graph.

Keywords – Graceful Labeling, Path Union, Grid Graph, Super Subdivision of Graph, Tensor Product of Two Graphs.

AMS Subject Classification Number: 05C78

I. INTRODUCTION

The graceful labeling was introduced by A. Rosa [1] during 1967. Acharya and Gill [2] have investigated graceful labeling for the grid graph $P_m \times P_n$. Kaneria and Makadia [3] discussed gracefulness of union of two grids, tensor product of $P_3$ and $P_n$, star of cycle $C_n$ ($n \equiv 0 \pmod{4}$). Kaneria and et al. [4] discussed gracefulness of $C_m$ ($m \equiv 0 \pmod{4}$) and wheel $W_n$ with a path of arbitrary length is graceful. For a dynamic survey on graph labeling one can refer Gallian[5].

We shall begin with a simple graph $G = (V, E)$, which is a finite, undirected graph with $|V|$ = $p$ vertices and $|E|$ = $q$ edges. For all terminology and notation we follows Harary [6]. First we shall recall some basic definitions which are used in this paper.

Definition – 1.1: A function $f$ is called graceful labeling of a graph $G = (V, E)$ if it is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e) = |f(u) – f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph $G$ is called graceful graph if it admits a graceful labeling.

Definition – 1.2: The Cartesian product of graphs $G_1$ and $G_2$, denoted as $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2) = \{(u,v) / u \in V(G_1) \text{ and } v \in V(G_2)\}$ and $(u,v)$ adjacent to $(u',v')$ if and only if either $u = u'$ and $(v,v') \in E(G_2)$ or $v = v'$ and $(u,u') \in E(G_1)$.

The Cartesian product of two paths $P_m$ and $P_n$, denoted as $P_m \times P_n$, is known as a grid graph on $mn$ vertices and $2mn – (m+n)$ edges.

Definition – 1.3: The Tensor product of two graphs $G_1$ and $G_2$ denoted by $G_1 \otimes G_2$ is a graph with vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \otimes G_2) = \{(u_1, v_1), (u_2, v_2) / (u_1, u_2) \in E(G_1) \text{ and } (v_1, v_2) \in E(G_2)\}$.

Definition – 1.4: Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be a $t$–super subdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e$ of $G$ by a complete bipartite graph $K_{2,t}$, for some $t \in N$.

In this paper we have proved $t$–super subdivision of the grid graph $P_m \times P_n$ and the tensor product of $P_m$ and $P_n$ (when $m, n$ both are odd) are graceful graphs. Also by joining $C_p$ ($p \equiv 0 \pmod{4}$) and a grid graph $P_m \times P_n$ with a path of arbitrary length $t$ is graceful graph.

II. MAIN RESULTS

Theorem – 2.1: $t$ – super subdivision of $P_m \times P_n$, where $m, n \in N – \{1\}$ is graceful graph.

Proof: Let $v_{ij} \ (1 \leq i \leq n, 1 \leq j \leq m)$ be vertices of the grid graph $P_m \times P_n$. We know that the number of vertices in $P_m \times P_n$ is $p = mn$ and the number of edges $q = 2mn – (m+n)$.

Let $G$ be a graph obtained by $t$– super subdivision of $P_m \times P_n$. Then we see that the number of vertices in $G$ is $P = \left|V(G)\right| = p + tq$ and the number of edges in $G$ is $Q = \left|E(G)\right| = 2tq$.

Let $u_{i,j,k} (i \leq j \leq n)$ be vertices for the vertical edges in $G$, and $w_{i,j,k} (i \leq j \leq m)$ be vertices for horizontal edges in $G$.

We define the labeling function $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows:

$f(v_{i,j}) = Q - \left[ n(j-1) + (i-1) \right] + \left[ \left(3n-2\right)(j-1)+（i-1)\right] + (k-1)$, where $i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, m, k = 1, 2, \ldots, t$.

$f(w_{i,j,k}) = \left[ \left(3n-2\right)(j-1)+（j-1)\right] - k$,

where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m-1, k = 1, 2, \ldots, t$.

The above labeling pattern give rise graceful labeling to the graph $G$ obtained by $t$– super subdivision of the grid graph $P_m \times P_n$ and $G$ is a graceful graph.

Illustration – 2.2: $4$ – super subdivision of grid graph $P_3 \times P_3$ and its graceful labeling shown in figure –1, where $p=9, q=12, P=57, Q=96$.
Theorem 2.3: \( P_m ( T_p ) P_n \) is graceful, when \( m, n = 1 \) (mod 2) and \( m, n \in N \setminus \{ 1 \} \). Where \( P_m \) and \( P_n \) are paths on \( m \) and \( n \) vertices respectively.

Proof:

Let \( G \) be the graph obtained by the tensor product of two paths \( P_m \) and \( P_n \) of odd lengths.

Let \( v_{i,j} \) ( \( 1 \leq i \leq m \), \( 1 \leq j \leq n \) ) be vertices of the be vertices of the graph \( G = P_m ( T_p ) P_n \), where \( m, n = 1 \) (mod 2) and \( m, n \in N \setminus \{ 1 \} \). We define the labeling function \( f : V(G) \rightarrow \{0, 1, \ldots, q\} \), where \( q = 2(m-1)(n-1) \) as follows:

\[
f(v_{i,1}) = \frac{i-1}{2}, \quad \forall i = 1, 3, 5, \ldots, m;
\]

\[
f(v_{i,j}) = f(v_{i-1,j}) + (m-1), \quad \forall i = 1, 3, 5, \ldots, m; \quad j = 3, 5, 7, \ldots, n;
\]

\[
f(v_{1,j}) = f(v_{m,j}) + \left( \frac{m-1}{2} \right) - \left( \frac{i-1}{2} \right), \quad \forall i = 1, 3, 5, \ldots, m;
\]

\[
f(V_{i,j}) = f(V_{i,j-2}) - (m-1), \quad \forall j = 4, 6, \ldots, n-1;
\]

\[
f(V_{1,j}) = q - \left( \frac{j-2}{2} \right), \quad \forall i = 2, 4, 6, \ldots, m-1;
\]

\[
f(V_{i,j}) = f(V_{i,j-2}) - (m-1), \quad \forall i = 2, 4, \ldots, m-1;
\]

\[
f(V_{i,n}) = f(V_{i,n-1}) + \frac{i}{2}, \quad \forall i = 2, 4, \ldots, m-1;
\]

\[
f(V_{i,j}) = f(V_{i,j+2}) - (m-1), \quad \forall j = n-2, n-4, \ldots, 1.
\]

The above labeling pattern give rise graceful labeling to the graph \( G \), obtained by the tensor product of two paths of odd lengths.

Therefore \( G = P_m ( T_p ) P_n \), where \( m, n = 1 \) (mod 2) and \( m, n \in N \setminus \{ 1 \} \) is graceful graph.

Illustration 2.4: \( P_3 ( T_1 ) P_7 \) and its graceful labeling shown in figure 2, where \( p=35 \) and \( q=48 \).

Theorem 2.5: A graph obtain by joining \( C_p \) ( \( p = 0 \) (mod 4)) and a grid graph \( P_n \times P_m \) with a path of arbitrary length \( t \) is graceful.

Proof:

Let \( G \) be the graph obtained by joining a cycle \( C_p \) ( \( p = 0 \) (mod 4)) and a grid graph \( P_n \times P_m \) with \( p, \) a path of length \( t \) on \( t+1 \) vertices.

Let \( u_1, u_2, u_3, \ldots, u_p \) be the vertices of \( C_p \), \( v_1 = u_p, v_2, \ldots, v_{t+1} \) be vertices of path \( P_t \) of \( t \) length and \( w_{1,1}, w_{1,2}, \ldots, w_{1,m}, w_{2,m}, \ldots, w_{n,1}, \ldots, w_{n,m} \) be the vertices of the grid graph \( P_n \times P_m \) where \( v_{t+1} = w_{1,1} \).

We know that the grid graph \( P_n \times P_m \) is graceful graph on \( p=mn \) vertices and \( q=2mn-(m+n) \) edges. Let \( f : V(P_n \times P_m) \rightarrow \{0, 1, \ldots, q\} \) be the graceful labeling with two sequence of labels, among one is increasing and another is decreasing, which start by \( f(w_{1,1}) = q \) and end with \( f(w_{n,m}) = \left\lfloor \frac{q}{2} \right\rfloor \).

To define labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \), where \( Q = q + t + p = p + t + 2mn - (m+n) \), we shall take following two cases.

Case 1: \( t \) is odd

\[
g(u_i) = Q + \left( \frac{i-1}{2} \right), \quad \forall i = 1, 3, 5, \ldots, p - 1
\]

\[
= \frac{i-2}{2}, \quad \forall i = 2, 4, 6, \ldots, \frac{p}{2};
\]

\[
g(v_i) = Q + 1 - \left( \frac{p+i}{2} \right), \quad \forall i = 2, 4, \ldots, t + 1
\]

\[
= \frac{p+i-1}{2}, \quad \forall i = 1, 3, \ldots, t;
\]

\[
g(w) = f(w) + \frac{p+t+1}{2}, \quad \forall w \in v(p_n \times p_m).
\]

Case 2: \( t \) is even

\[
g(u_i) = Q - \left( \frac{i+1}{2} \right), \quad \forall i = 1, 3, 5, \ldots, p - 1
\]
Above labeling pattern give rise graceful labeling to the given graph $G$.

**Illustration – 2.6**: A cycle $C_{12}$, a grid graph $P_3 \times P_3$ joining by a path $P_5$ of length with its graceful labeling shown in figure –3, where $q=12$, $t=5$, $P = 25$, $Q = q + t + p = 29$.

Fig. 3. A cycle $C_{12}$, a grid graph $P_3 \times P_3$ joining by a path $P_5$ with its graceful labeling.

**III. CONCLUDING REMARKS**

Here we have discussed the gracefulfulness of $t$–super subdivision of $(P_q \times P_m)$ and $P_m (T_P) P_n$, the tensor product of two paths of odd lengths. We also discussed gracefulfulness of a cycle $C_p (p \equiv 0 \mod 4)$ joining with a grid graph $P_n \times P_m$ by a path $P_t$ of arbitrary length $t$. This work contributes three new results to the families of graceful labeling. The labeling pattern is demonstrated by means of illustrations.

**REFERENCES**


