Fuzzy Translation and Fuzzy Multiplication on PS-Algebras

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Abstract – In this paper, we define a fuzzy translation and fuzzy multiplication on PS-algebras and discussed some of their properties in detail by using the concepts of fuzzy PS-ideal and fuzzy PS-sub algebra.

Keywords – Fuzzy-α-Translation, Fuzzy-α-Multiplication of Fuzzy PS-Algebra, Fuzzy PS-Ideal, Fuzzy PS-Sub Algebra, Fuzzy Extension.

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I. INTRODUCTION

The concept of fuzzy set was initiated by L. A. Zadeh in 1965 [4]. It has opened up keen insights and applications in a wide range of scientific fields. Since its inception, the theory of fuzzy subsets has developed in many directions and found applications in a wide variety of fields. The study of fuzzy subsets and its applications to various mathematical contexts has given rise to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. Fuzzy ideas have been applied to other algebraic structures such as groups, rings, modules, vector spaces and topologies. In this way, K. Iséki and S. Tanaka [1] introduced the concept of BCK-algebras in 1978. K. Iséki [2] introduced the concept of BCI-algebras in 1980. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. T. Priya and T. Ramachandran [6][7] introduced the class of PS-algebras, which is a generalization of BCI / BCK/Q / KU / δ algebras. In this paper, we introduce the concept of fuzzy-α-translation, fuzzy-α-multiplication of fuzzy PS-algebras and fuzzy extensions and established some of its properties in detail.

II. PRELIMINARIES

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1 [1]
A BCK- algebra is an algebra (X,* ,0) of type(2,0) satisfying the following conditions:

i) (x * y) * (x * z) ≤ (z * y)

ii) x * (x * y) ≤ y

iii) x ≤ x

iv) x ≤ y and y ≤ x ⇒ x = y

v) 0 ≤ x ⇒ x = 0, where x ≤ y is defined by x * y = 0 , for all x, y, z ∈ X.

Definition 2.2 [2]
A BCI- algebra is an algebra (X,* ,0) of type(2,0) satisfying the following conditions:

i) (x * y) * (x * z) ≤ (z * y)

ii) x * (x * y) ≤ y

iii) x ≤ x

iv) x ≤ y and y ≤ x ⇒ x = y

v) 0 ≤ x ⇒ x = 0, where x ≤ y is defined by x * y = 0 , for all x, y, z ∈ X.

Definition 2.3 [7]
A nonempty set X with a constant 0 and a binary operation ‘·’ is called PS – algebra if it satisfies the following axioms.
1. x • x = 0
2. x • 0 = 0
3. x • y = 0 and y • x = 0 ⇒ x = y , ∀ x, y ∈ X.

Definition 2.4 [7]
Let X be a PS-algebra and I be a subset of X, then I is called a PS-ideal of X if it satisfies following conditions:
1. 0 ∈ I
2. x • y ∈ I and y ∈ I ⇒ x ∈ I

Definition 2.5 [7]
Let X be a PS-algebra. A fuzzy set μ in X is called a fuzzy PS-ideal of X if it satisfies the following conditions.
1. μ(0) ≥ μ(x)
2. μ(μ(x)) ≥ μ(x)
3. μ(μ(y), μ(y)), for all x, y ∈ X

Definition 2.6 [6]
A fuzzy set μ in a PS-algebra X is called a fuzzy PS-sub algebra of X if μ(x * y) ≥ min {μ(x), μ(y)}, for all x, y ∈ X.

III. FUZZY TRANSLATION AND FUZZY MULTIPLICATION IN PS-ALGEBRA

Let X be a PS-algebra. For any fuzzy set μ of X, we define T = 1− sup {μ(x) / x ∈ X }, unless otherwise we specified.

Definition 3.1 [(3)[5]]
Let μ be a fuzzy subset of X and α ∈ [0, T]. A mapping μ_α^T : X → [0, 1] is said to be a fuzzy-α-translation of μ if it satisfies μ_α^T (x) = μ(x) + α , ∀ x ∈ X.

Definition 3.2 [(3)[5]]
Let μ be a fuzzy subset of X and α ∈ [0, 1]. A mapping μ_α^M : X → [0, 1] is said to be a fuzzy-α-multiplication of μ if it satisfies μ_α^M (x) = α μ(x) , ∀ x ∈ X.

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Choose \( \mu \) be a fuzzy subset of \( X \) such that \( \mu \) is a fuzzy PS-sub algebra of \( X \). Here \( T = 1 - \sup \{ \mu(x) / x \in X \} = 1 - 0.3 = 0.7 \). Choose \( \alpha = 0.4 \in [0, T] \) and \( \beta = 0.5 \in [0, 1] \).

Then the mapping \( \mu_{0.4} : X \rightarrow [0, 1] \) is defined by
\[
\mu_{0.4}(x) = \begin{cases} 0.3 & \text{if } x \neq 1 \\ 0.2 & \text{if } x = 1 
\end{cases}
\]
which satisfies \( \mu_{0.4} \) of \( \mu \) is a fuzzy PS-sub algebra of \( X \). Define a fuzzy set \( \mu \) be a fuzzy subset of \( X \) such that \( \mu \) is a fuzzy PS-sub algebra of \( X \). Here \( T = 1 - \sup \{ \mu(x) / x \in X \} = 1 - 0.3 = 0.7 \). Choose \( \alpha = 0.4 \in [0, T] \) and \( \beta = 0.5 \in [0, 1] \).

Then the mapping \( \mu_{0.4} : X \rightarrow [0, 1] \) is defined by
\[
\mu_{0.4}(x) = \begin{cases} 0.3 + 0.4 = 0.7 & \text{if } x \neq 1 \\ 0.2 + 0.4 = 0.6 & \text{if } x = 1 
\end{cases}
\]
which satisfies \( \mu_{0.4} \) of \( \mu \) is a fuzzy PS-sub algebra of \( X \). Theorem 3.4:

If \( \mu \) of \( X \) is a fuzzy PS-sub algebra and \( \alpha \in [0, T] \), then the fuzzy-\( \alpha \)-translation \( \mu_{\alpha} \) of \( \mu \) is also a fuzzy PS-sub algebra of \( X \).

Proof:
Let \( x, y \in X \) and \( \alpha \in [0, T] \).
Then \( \mu(x * y) \geq \min \{ \mu(x), \mu(y) \} \)
Now \( \mu_{\alpha}(x * y) = \mu(x * y) + \alpha \)
\[
\geq \min \{ \mu(x), \mu(y) \} + \alpha \\
= \min \{ \mu(x) + \alpha, \mu(y) + \alpha \} \\
= \min \{ \mu_{\alpha}(x), \mu_{\alpha}(y) \}
\]
Theorem 3.5:
Let \( \mu \) be a fuzzy subset of \( X \) such that the fuzzy-\( \alpha \)-translation \( \mu_{\alpha} \) of \( \mu \) is a fuzzy PS-sub algebra of \( X \) for some \( \alpha \in [0, T] \), then \( \mu \) is a fuzzy PS-sub algebra of \( X \).

Proof:
Assume that \( \mu_{\alpha} \) is a fuzzy subalgebra of \( X \) for some \( \alpha \in [0, T] \).
Let \( x, y \in X \). We have
\[
\mu(x * y) + \alpha = \mu_{\alpha}(x * y) \\
\geq \min \{ \mu_{\alpha}(x), \mu_{\alpha}(y) \} \\
= \min \{ \mu(x) + \alpha, \mu(y) + \alpha \} \\
= \min \{ \mu_{\alpha}(x), \mu_{\alpha}(y) \} + \alpha \\
\Rightarrow \mu(x * y) \geq \min \{ \mu(x), \mu(y) \} \text{ for all } x, y \in X. \text{ Hence } \mu \text{ is fuzzy subalgebra of } X.
\]
Theorem 3.6:
For any fuzzy PS-sub algebra \( \mu \) of \( X \) and \( \alpha \in [0, T] \), if the fuzzy-\( \alpha \)-multiplication \( \mu_{\alpha} \) of \( \mu \) is a fuzzy PS-sub algebra of \( X \).

Proof:
Let \( x, y \in X \) and \( \alpha \in [0, T] \).
Then \( \mu(x * y) \geq \min \{ \mu(x), \mu(y) \} \)
Now,
\[
\mu_{\alpha}(x * y) = \alpha \mu(x * y) \\
\geq \alpha \min \{ \mu(x), \mu(y) \} \\
= \min \{ \alpha \mu(x), \alpha \mu(y) \} \\
= \min \{ \mu_{\alpha}(x), \mu_{\alpha}(y) \} \\
\Rightarrow \mu_{\alpha}(x * y) \geq \min \{ \mu_{\alpha}(x), \mu_{\alpha}(y) \} \\
\Rightarrow \mu_{\alpha} \text{ is a fuzzy PS-sub algebra of } X.
\]
Also, \( \mu(x) + \alpha = \mu_T(x) \)
\[ \geq \min \{ \mu_T(y^*x) , \mu_T(y) \} \]
\[ = \min \{ \mu(y^*x) + \alpha , \mu(y) + \alpha \} \]
and so \( \mu(x) \geq \min \{ \mu(y^*x) , \mu(y) \} \)
Hence \( \mu \) is a fuzzy PS-ideal of X.

**Theorem 3.11:**
Let \( \alpha \in [0,T] \) and let \( \mu \) be a fuzzy PS-ideal of X. If X is a PS-algebra, then the fuzzy \( \alpha \) - translation \( \mu_T \) of \( \mu \) is a fuzzy PS-sub algebra of X.

**Proof:**
Let \( x, y \in X \).
Now, we have\n\[ \mu_T(x*y) = \mu_T(x) + \alpha \]
\[ \geq \min \{ \mu_T(y^*x) , \mu_T(y) \} \]
\[ = \min \{ \mu(y^*x) + \alpha , \mu(y) + \alpha \} \]
\[ = \min \{ \mu_T(x) , \mu_T(y) \} \]
Hence \( \mu_T \) is a fuzzy PS-sub algebra of X.

**Theorem 3.12:**
If the fuzzy \( \alpha \) - translation \( \mu_T \) of \( \mu \) is a fuzzy PS-ideal of X, \( \alpha \in [0,T] \), then \( \mu \) is a fuzzy PS-sub algebra of X.

**Proof:**
Let us assume that \( \mu_T \) of \( \mu \) is a fuzzy PS-ideal of X. Then
\[ \mu(x*y) + \alpha = \mu_T(x*y) \]
\[ \geq \min \{ \mu_T(y^*x) , \mu_T(y) \} \]
\[ = \min \{ \mu_T(x) , \mu_T(y) \} \]
\[ \geq \mu(x) \]
Hence \( \mu \) is a fuzzy PS-sub algebra of X.

**Theorem 3.13:**
Let \( \mu \) be a fuzzy subset of X such that the fuzzy \( \alpha \)-multiplication \( \mu^M \) of \( \mu \) is a fuzzy PS-ideal of X for some \( \alpha \in (0,1] \), then \( \mu \) is a fuzzy PS-ideal of X.

**Proof:**
Assume that \( \mu^M \) is a fuzzy PS-ideal of X for some \( \alpha \in (0,1] \). Let \( x, y \in X \).
Then \( \alpha \mu(0) = \mu^M(0) \)
\[ \geq \mu^M(x) \]
and so \( \mu(0) \geq \mu(x) \)
Also, \( \alpha \mu(x) = \mu^M(x) \)
\[ \geq \min \{ \mu^M(y^*x) , \mu^M(y) \} \]
\[ = \min \{ \alpha \mu(y^*x) , \alpha \mu(y) \} \]
and so \( \mu(x) \geq \min \{ \mu(y^*x) , \mu(y) \} \)
Hence \( \mu \) is a fuzzy PS-ideal of X.

**Theorem 3.14:**
If \( \mu \) is a fuzzy PS-ideal of X, then the fuzzy \( \alpha \)-multiplication \( \mu^M \) of \( \mu \) is a fuzzy PS-ideal of X, for all \( \alpha \in (0,1] \).

**Proof:**
Let \( \mu \) be a fuzzy PS-ideal of X and let \( \alpha \in (0,1] \).

Then
\[ \mu_T(0) = \alpha \mu(0) \]
\[ \geq \alpha \mu(x) \]
\[ = \mu^M(x) \]
And \( \mu_T(x) = \alpha \mu(x) \)
\[ \geq \alpha \mu(y^*x) \]
\[ = \mu^M(x) \]
\[ = \mu_T(x) \]
Hence \( \mu_T \) and \( \mu_T \) are fuzzy PS-ideals of X.
IV. Fuzzy Extensions of PS-Ideals of PS-Algebras

In this section, we introduced the of fuzzy extensions of PS-ideals of PS-algebras and proved some standard results.

Definition 4.1:
Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy sets of \( X \) such that \( \mu_2 \) is a fuzzy extension of \( \mu_1 \). If \( \mu_1 \) is a fuzzy PS-ideal of \( X \) implies that \( \mu_2 \) is a fuzzy PS-ideal of \( X \), then \( \mu_2 \) is called as fuzzy PS-ideal extension of \( \mu_1 \).

Example 4.2: Consider the PS-algebra as in example 3.3, the fuzzy sets \( \mu_1 \) and \( \mu_2 \) of \( X \) is defined as follows:

\[
\mu_1(x) = \begin{cases} 
0.7 & \text{if } x = 0 \\
0.5 & \text{if } x \neq 0 
\end{cases}
\quad \text{and} \quad \mu_2(x) = \begin{cases} 
0.8 & \text{if } x = 0 \\
0.7 & \text{if } x = 1 \\
0.5 & \text{if } x = 2
\end{cases}
\]

are fuzzy PS-ideals of \( X \). It is clear that \( \mu_2 \) is a fuzzy PS-ideal extension of \( \mu_1 \).

Theorem 4.3:
Intersection of any two fuzzy PS-ideal extensions of a fuzzy PS-ideal \( \mu \) of \( X \) is a fuzzy PS-ideal extension of \( \mu \).

Proof:
Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy PS-ideal extensions of a fuzzy PS-ideal \( \mu \) of \( X \). Then \( \mu_1(x) \geq \mu(x) \) and \( \mu_2(x) \geq \mu(x) \), for all \( x \in X \). Since \( \mu \) is a fuzzy PS-ideal of \( X \), \( \mu_1 \) and \( \mu_2 \) are fuzzy PS-ideals of \( X \). Then \( \mu_1 \cup \mu_2 \) is also a fuzzy PS-ideal of \( X \)(By theorem 3.4(6)).

Here \( (\mu_1 \cup \mu_2)(x) = \min \{\mu_1(x), \mu_2(x)\} \geq \min \{\mu_1(x), \mu_2(x)\} = \mu(x) \). Hence \( \mu_1 \cup \mu_2 \) is a fuzzy PS-ideal extension of \( \mu \).

Remark:
Union of any two fuzzy PS-ideal extensions of a fuzzy PS-ideal \( \mu \) of \( X \) need not be a fuzzy PS-ideal extension of \( \mu \).

Consider the example 3.3, the fuzzy sets \( \mu \), \( \mu_1 \) and \( \mu_2 \) of \( X \) is defined as follows:

\[
\mu(x) = \begin{cases} 
0.6 & \text{if } x = 0 \\
0.5 & \text{if } x \neq 0 
\end{cases}
\quad \text{and} \quad \mu_1(x) = \begin{cases} 
0.7 & \text{if } x = 0 \\
0.5 & \text{if } x \neq 0 
\end{cases}
\quad \text{and} \quad \mu_2(x) = \begin{cases} 
0.8 & \text{if } x = 0 \\
0.7 & \text{if } x = 1 \\
0.5 & \text{if } x = 2
\end{cases}
\]

From the above definition itself it is clear that \( \mu_1 \) and \( \mu_2 \) are fuzzy PS-ideal extensions of the fuzzy PS-ideal \( \mu \). Here \( (\mu_1 \cup \mu_2)(0) = \max \{\mu_1(0), \mu_2(0)\} = 0.8 \) ; \( (\mu_1 \cup \mu_2)(1) = 0.7 \) ; \( (\mu_1 \cup \mu_2)(2) = 0.5 \), is not a fuzzy PS-ideal extension, since \( (\mu_1 \cup \mu_2)(2) = 0.5 \geq 0.7 = \min \{(\mu_1 \cup \mu_2)(1*2), (\mu_1 \cup \mu_2)(1)\} = \min\{(\mu_1 \cup \mu_2)(1), (\mu_1 \cup \mu_2)(1)\} \).

Theorem 4.4:
Let \( \mu \) be a fuzzy PS-ideal of \( X \). The fuzzy \( \alpha \)- translation \( \mu^\alpha \) is a fuzzy PS-ideal extension of \( \mu \), for all \( \alpha \in [0,1] \).

Proof:
If \( \mu \) is a fuzzy PS-ideal of \( X \), then by theorem 3.11, the fuzzy \( \alpha \)-translation \( \mu^\alpha \) of \( \mu \) is also a fuzzy PS-ideal of \( X \), for all \( \alpha \in [0,1] \). Now \( \mu^\alpha x \in X \). Hence, the fuzzy \( \alpha \) - translation \( \mu^\alpha \) is a fuzzy PS-ideal extension of \( \mu \).

Theorem 4.5:
Let \( \mu \) be a fuzzy PS-ideal of \( X \). If \( \alpha \leq \delta \), with \( \alpha, \delta \in [0,1] \), then the fuzzy \( \alpha \)-translation \( \mu^\alpha \) of \( \mu \) is a fuzzy PS-ideal extension of the fuzzy \( \delta \)-translation \( \mu^\delta \) of \( \mu \).

Proof:
Let \( \mu \) be a fuzzy PS-ideal of \( X \). Then by theorem 3.11, the fuzzy \( \alpha \)-translation \( \mu^\alpha \) of \( \mu \) and the fuzzy \( \delta \)-translation \( \mu^\delta \) of \( \mu \) are fuzzy PS-ideals of \( X \), for all \( \alpha, \delta \in [0,1] \). Therefore \( \mu^\alpha \) and \( \mu^\delta \) is a fuzzy PS-ideal extension of \( \mu^\delta \).

Theorem 4.6:
Let \( \mu \) be a fuzzy PS-ideal of \( X \). Then by theorem 3.11, the fuzzy \( \alpha \)-translation \( \mu^\alpha \) of \( \mu \) and the fuzzy \( \delta \)-translation \( \mu^\delta \) of \( \mu \) are fuzzy PS-ideals of \( X \), for all \( \alpha, \delta \in [0,1] \). Therefore \( \mu^\alpha \) and \( \mu^\delta \) is a fuzzy PS-ideal extension of \( \mu^\delta \).

V. Conclusion

In this article authors have been discussed fuzzy translation and fuzzy multiplication on PS-algebras through PS-sub algebras and PS-ideals. It has been observed that PS-algebras as an another generalization of BCK/BCI/Q/d/TM/KU algebras. Interestingly, fuzzy extensions of PS-ideals of PS-algebras has been studied, which adds another dimension to the defined PS-algebras. This concept can further be generalized to Intuitionistic fuzzy set, interval valued fuzzy sets for new results in our future work.

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