International Journal of Innovation in Science and Mathematics
Volume 3, Issue 3, ISSN (Online): 2347–9051

New Type of Non Differentiable Second order Symmetric Dual Model for Multi-Objective Programs under Generalized F-Convexity

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Abstract **– New type of non differentiable higher order duality models are introduced. Weak duality and strong duality results are established. Special cases are discussed.**

Keywords **– Second Order F Convexity, Multi Objective Programming, Support Functions, Symmetric Duality.**

I. Introduction

In mathematical program, a pair of primal and dual program is called symmetric if the dual of the dual is the primal problem. The duality in linear programming is symmetric. The first symmetric dual formulation for quadratic programming was proposed by Dorn [1]. Mond [2] initated second order symmetric duality of wolfe type in nonlinear programming and proved the duality theorems under second order convexity. Mangasarian [3] discussed second order duality in non linear programming under inclusion condition. This motivated several authors like Bector and Chandra [4], Mond and Weir [5], Devi [6], Gulati etal. [7] in this field. Also Yang et al [7, 8] formulated a pair of wolf type second order non differentiable symmetric dual programs containing support function under F-convexity.

Recently, Gulati and Geeta [10] studied mond-weir type second order symmetric duality in multiobjective programming over cones and established duality results under F convexity assumption.

In this paper, we introduce two models of mixed symmetric duality for a class of second order non differentiable multi-objective programming problems with multiple arguments. Mixed symmetric duality for new model has not been given so far by any other author. We establishe weak and strong duality theorems for these models and discuss several special cases of these models with support functions.

II. Preliminaries and Notation

The following convention for vectors in R^n will be used;
$x < y \Leftrightarrow x_i < y_i, i = 1,2,..n$, $x \leqq y \Leftrightarrow x_i \leqq y_i$, $i = 1,2,...,n$ and $x \le y \Leftrightarrow x_i \le y_i$, $i = 1, 2, \ldots, n$ but, $x \ne y$

Definition 2.1: A set C of R^n is called a cone if for each $x \in C$ and $\lambda \in R$, $\lambda \ge 0$ we have $\lambda x \in C$. Moreover, if C is convex, then it is the convex cone.

Definition 2.2 : The positive polar cone C^* of C is defined as $C^* = \{z \in R^n \mid x^T z \ge 0, \forall x \in C\}$.

Definition 2.3 : Let C be a compact convex set in R^n. The support functions is defined by

$$s(x \mid c) = \max\{x^T y : y \in C\}$$

A support function, being convex and everywhere finite, has a sub-differential, that is there exists $z \in R^n$ such that

$$s(y \mid c) \ge s(x \mid c) + z^T(y - x),\ \forall y \in C$$

The sub differential of $s(x \mid c)$ is given by

$$\partial s(x \mid c) = \{z \in C : z^T x = s(x \mid c)\}$$

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T(z - x) \le 0,\ \forall z \in D\}$$

It is obvious that for a compact convex set C, $y \in N_c(x)$ if and only if $s(y \mid c) = x^T y$, or equivalently, $x \in \partial s(y \mid c)$.

The following definitions will be needed in the sequal.

Definition 2.4: Let $x \subset R^n$. A functional $F : X \times X \times R^n \to R$ is said to be sublinear with respect to its third argument if for any $x, y \in X$.

(A) $F(x, y; a_1 + a_2) \le F(x, y; a_1) + F(x, y; a_2)$ for any $a_1, a_2 \in R^n$.

(B) $F(x, y; da) \le \alpha F(x, y; a)$ for any $\alpha \in R_+$ and $a \in R^n$

Definition 2.5: Let $X \subset R^n$, $y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. f (., y) is said to be F-convex at $\bar{x} \in X$ for fixed $y \in Y$ if

$$f(x, y) - f(\bar{x}, y) \ge F(x, \bar{x}; \nabla_x f(\bar{x}, y)),\ \forall x \in X$$

Definition 2.6: A twice differentiable function $f = (f_1, f_2, \ldots, f_k) : x \to R^k$ is said to be second order F convex at $\bar{x} \in X$, for fixed $y \in Y$ and sublinear functional $F : X \times X \times R^n \to R$, such that for all $(x, \bar{x}, p) \in X \times X \times R^n$, we have

IJISM Source of Knowledge
International Journal of Innovation in Science and Mathematics
Volume 3, Issue 3, ISSN (Online): 2347–9051

$$\left[f_i(x,y)-f_i(\bar{x},y)+\frac{1}{2}p_i^T\nabla_{xx}f_i(\bar{x},y)p_i\right]\geq F\left(x,\bar{x};\left\{\nabla_x f_i(\bar{x},y)+\nabla_{xx}f_i(\bar{x},y)p_1\right\}\right)$$

Definition 2.7: A twice differentiable function $f:X\to R^k$ is said to be second order F-pseudo convex at $\bar{x}\in X$, for fixed $y\in Y$ and sublinear functional $F:X\times X\times R^n\to R$, such that for all $(x,\bar{x},p)\in X\times X\times R^n$, we have

$$F\left(x,\bar{x};\left\{\nabla_x f_1(\bar{x},y)+\nabla_{xx}f_1(\bar{x},y)p_1\right\}\right)\geq 0$$

$$\Rightarrow f_1(x,y)-f_1(\bar{x},y)+\frac{1}{2}p_1^T\nabla_{xx}f_1(\bar{x},y)p_1\geq 0$$

III. Mixed Type Second Order Symmetric Duality

For N = {1, 2, ….., n} and M = {1, 2, …..,m}, $P_1\subset N$, $L_1\subset m$ and $P_2=N\setminus p_1$ and $L_2=m\setminus R_1$

Let $|P_1|$ denote the number of elementrs in set P_1 and $|P_2|$ denote the number of elements in the set P_2.

It is clear that any $x\in R^n$ can be written as $x=(x^1,x^2), x^1\in R^{|p_1|}, x^2\in R^{|p_2|}$

Similarly any $y\in R^m$ can be written as

$$y=(y^1,y^2),\ y^1\in R^{|L_1|},\ y^2\in R^{|L_2|}$$

Let $f:R^{|P_1|}\times R^{|L_1|}\to R^k$ and $g:R^{|P_2|}\times R^{|L_2|}\to R^k$ be twice differentiable functions and $e=[1,1,\ldots 1]^T\in R^k$

Primal Problem (MHP)

Wolfe type second order multi objective non differentiable symmetric dual programs.

$$\text{Min } H(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)$$
$$=H_1(x^1,x^2,y^1,y^2,z^1,z^2,\lambda),\ldots,H_k(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)$$

Subject to $(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)\in R^{|p_1|}\times R^{|p_2|}\times R^{|L_1|}\times R^{|L_2|}\times R^{|L_1|}\times R^{|L_2|}\times R_+^k$

$$\sum_{i=1}^k\lambda_i\left[\nabla_y f_i(x^1,y^1)+\nabla_{yy}f_i(x^1,y^1)p-z_i^1\right]\leq 0 \quad (3.1)$$

$$\sum_{i=1}^k\lambda_i\left[\nabla_y g_i(x^2,y^2)+\nabla_{yy}g_i(x^2,y^2)p-z_i^2\right]\leq 0 \quad (3.2)$$

$$(y^1)^T\sum_{i=1}^k\lambda_i\left[\nabla_y f_i(x^1,y^1)+\nabla_{yy}f_i(x^1,y^1)p-z_i^1\right]\geq 0 \quad (3.3)$$

$$(y^2)^T\sum_{i=1}^k\lambda_i\left[\nabla_y g_i(x^2,y^2)+\nabla_{yy}g_i(x^2,y^2)p-z_i^2\right]\geq 0 \quad (3.4)$$

$$(x^1,x^2)\geq 0$$

$$z_i^1\in D_i^1 \text{ and } z_i^2\in D_i^2,\ i=1,2,\ldots.k$$

$$\lambda>0,\ \sum_{i=1}^k\lambda_i=1 \quad (3.5)$$

Dual Problem (MHD)

$$\text{Max} G(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)$$
$$=\left(G_1(u^1,u^2,v^1,v^2,w^1,w^2,\lambda),\ldots\ldots,G_k(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)\right)$$

Subject to

$$(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)\in R^{|p_1|}\times R^{|p_2|}\times R^{|L_1|}\times R^{|L_2|}\times R^{|L_1|}\times R^{|L_2|}\times R_+^k$$

$$\sum_{i=1}^k\lambda_i\left[\nabla_{u^1}f_i(u^1,v^1)+\nabla_{u^1u^1}f_i(u^1,v^1)p+w_i^1\right]\geq 0 \quad (3.6)$$

$$\sum_{i=1}^k\lambda_i\left[\nabla_{u^2}g_i(u^2,v^2)+\nabla_{u^2u^2}g_i(u^2,v^2)p+w_i^2\right]\geq 0 \quad (3.7)$$

$$(u^1)^T\sum_{i=1}^k\lambda_i\left[\nabla_{u^1}f_i(u^1,v^1)+\nabla_{u^1u^1}f_i(u^1,v^1)p+w_i^1\right]\leq 0 \quad (3.8)$$

$$(u^2)^T\sum_{i=1}^k\lambda_i\left[\nabla_{u^2}g_i(u^2,v^2)+\nabla_{u^2u^2}g_i(u^2,v^2)p+w_i^2\right]\leq 0 \quad (3.9)$$

$$(v^1,v^2)\geq 0, w_i^1\in C_i^1,\ w_i^2\in C_i^2,\ i=1,2,\ldots.,k \quad (3.10)$$

$$\lambda>0,\ \sum_{i=1}^k\lambda_i=1$$

where

$$H_i(x^1,x^2,y^1,y^2,z^1,z^2,\lambda)=f_i(x^1,y^1)+g_i(x^2,y^2)+s(x^1\mid C_i^1)+s(x^2\mid C_i^2)-(y^1)^T z_i^1-(y^2)^T z_i^2-\frac{1}{2}p^T\left(\nabla_{y^1y^1}f_i(x^1,y^1)+\nabla_{y^2y^2}g_i(x^2,y^2)\right)p$$

$$G_i(u^1,u^2,v^1,v^2,w^1,w^2,\lambda)=f_i(u^1,v^1)+g_i(u^2,v^2)-s(v^1\mid D_i^1)-s(v^2\mid D_i^2)+(u^1)^T w_i^1+(u^2)^T w_i^2-\frac{1}{2}\left[p^T\left(\nabla_{u^1u^1}f_i(u^1v^1)p+\nabla_{u^2u^2}g_i(u^2,v^2)p\right)\right]$$

Theorem 3.1 (Weak duality)

Let $\left(x^1,x^2,y^1,y^2,z^1,z^2,\lambda\right)$ be feasible for (MSP) and $\left(u^1,u^2,v^1,v^2,w^1,w^2,\lambda\right)$ be feasibly for (MSD).

Suppose $f_i\left(.,\ y^1\right)+.^T w_i^1$ is second order F_1-Pseudo convex for fixed $y^1, f_i\left(x^1,.\right)-.^T z_i^1$ is second order F_2 -pseudo concave for fixed x^1,

$g_i\left(.,\ y^2\right)+.^T w_i^2$ is second order G_1 – pseudo convex for fixed y^2 and $g_i\left(x^2,.\right)^T - .z_i^2$ is second order G_2-pseudo concave for fixed x^2 and the following conditions are satisfied.

(i) $F_1\left(x^1,u^1,a\right)+\left(u^1\right)^T a\ge 0,\ \text{if}\ a\ge 0$ (3.11)

(ii) $G_1\left(x^2,u^2,b\right)+\left(u^2\right)^T b\ge 0,\ \text{if}\ b\ge 0$ (3.12)

(iii) $F_2\left(y^1,v^1,c\right)+\left(y^1\right)^T c\le 0,\ \text{if}\ c\le 0$ and (3.13)

(iv) $G_2\left(y^2,v^2,d\right)+\left(y^2\right)^T d\le 0,\ \text{if}\ d\le 0$

Then $H_i\left(x^1,x^2,y^1,y^2,z,\lambda\right)\precneqq G_i\left(u^1,u^2,v^1,v^2,w,\lambda\right)$

Proof : Suppose $\left(x^1,x^2,y^1,y^2,z^1,z^2,\lambda\right)$ be feasible for (MSP) and $\left(u^1,u^2,v^1,v^2,w^1,w^2,\lambda\right)$ be feasible for (MSD)

Then by using second order of F_1-Pseudo convexity of $f_i\left(.,\ y^1\right)+.^T w_i^1$ and second order of F_2-pseudo concavity of $f_i\left(x^1,.\right)-.^T z_i^1$ for $i=1,\ 2,\ ..l$ we have

$$F_1\left[x^1,u^1;\left(\nabla_{x^1} f_i\left(u^1,y^1\right)+\nabla_{xx} f_i\left(u^1,y^1\right)+\left(u^1\right)^T w_i^1\right)\right]\ge 0$$

$$\Rightarrow\left[f_i\left(x^1,y^1\right)+x^{1T}w_i^1\right]\ge\left[f_i\left(u^1,v^1\right)+u^{1T}w_i^1\right] \quad (3.15)$$

and

$$F_2\left[v^1,y^1;\left(\nabla_{y^1} f_i\left(x^1,y^1\right)+\nabla_{yy} f_i\left(x^1,y^1\right)p-y^{1T}z_i^1\right)\right]\le 0$$

$$\Rightarrow\left[f_i\left(x^1,v^1\right)-\left(v^1\right)^T z_i^1\right]\le\left[f_i\left(x^1,y^1\right)-\left(y^1\right)^T z_i^1\right] \quad (3.16)$$

From equation 3.1, 3.6 and sublinearity of F_1 and F_2, we have

$$\sum_{i=1}^{k}\lambda_i\left[\nabla_y f_i\left(x^1,y^1\right)+\ \nabla_y f_i\left(x^1,y^1\right)p-z_i^1\right]\le 0$$

$$\sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i\left(u^1,v^1\right)+\ \nabla_u f_i\left(u^1,v^1\right)p-w_i^1\right]\ge 0$$

$$F_1\left(x^1,u^1;\nabla_{x^1} f_i\left(u^1,y^1\right)+\nabla_{xx} f_i\left(u^1,y^1\right)p\right)+\left(u^1\right)^T a\ge 0$$

where $a=\nabla_{x^1} f_i\left(u^1,y^1\right)+\nabla_{xx} f_i\left(u^1,y^1\right)p$

and

$$F_2\left(y^1,v^1;\nabla_{y^1} f_i\left(x^1,y^1\right)+\nabla_{yy} f_i\left(x^1,y^1\right)p\right)-\left(y^1\right)^T c\ge 0$$

where $c=\nabla_{y^1} f_i\left(x^1,y^1\right)+\nabla_{yy} f_i\left(x^1,y^1\right)p$

From equation (3.15) and (3.16), we get

$$\sum_{i=1}^{k}\lambda_i\left[f_i\left(x^1,y^1\right)+\left(x^1\right)^T w_i^1 f_i\left(u^1,v^1\right)-w_i^1\right]\ge 0$$

and

$$\sum_{i=1}^{k}\lambda_i\left[f_i\left(x^1,v^1\right)-\left(v^1\right)^T w_i^1-f_i\left(x^1,y^1\right)+\left(y^1\right)^T z_i^1\right]\le 0$$

Rearranging the above two inequalities, we obtain

$$\sum_{i=1}^{k}\lambda_i\left[f_i\left(x^1,y^1\right)-f_i\left(u^1-v^1\right)+\left(x^1\right)^T w_i^1-\left(u^1\right)^T w_i^1+\left(v^1\right)^T z_i^1-\left(y^1\right)^T z_i^1\right]\ge 0$$

Using $\left(v^1\right)^T z_i^1\le s\left(v^1\mid D_i^1\right)$ and $\left(x^1\right)^T w_i^1\le s\left(x^1\mid C_i^1\right)$, we have

$$\sum_{i=1}^{1}\lambda_i\left[f_i\left(x^1,y^1\right)-f_i\left(u^1,v^1\right)+s\left(x^1\mid c_i^1\right)-\left(u^1\right)^T w_i^1+s\left(v^1\mid D_i^1\right)-\left(y^1\right)^T z_i^1\right]\ge 0 \quad (3.17)$$

Similarly from $g_i\left(.,\ y^2\right)+.^T w_i^2$ is second order G_1 – pseudo convex for fixed y^2 and $g_i\left(x^2,.\right)-.^T z_i^2$ is second order G_2 – pseudo concave for fixed x^2 and constraint conditions 3.12 and 3.14, we get

$$\sum_{i=0}^{1}\lambda_i\left[g_i\left(x^2,y^2\right)+s\left(x^2\mid C_i^2\right)-\left(y^2\right)^T z_i^2-g_i\left(u^2,v^2\right)+s\left(v^2\mid D_i^2\right)-\left(u^2\right)^T w_i^2\right]\ge 0 \quad (3.18)$$

From equation (3.17) and (3.15) we get

$$H_i\left(x^1,x^2,y^1,y^2,z^1,z^2,\lambda\right)\precneqq G_i\left(u^1,u^2,v^1,v^2,w^1,w^2,\lambda\right)$$

Theorem 3.2 (Strong duality)

Let $\left(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{z}^1,\bar{z}^2,\lambda\right)$ be weakly efficient solution of (MSP) and $\bar{\lambda}=\lambda$ be fixed in (MSD) such that

(i) $\nabla_{yy}\left(\sum_{i=1}^{k}\lambda_i f_i\left(x^1,y^1\right)\right)$ and $\nabla_{yy}\left(\sum_{i=1}^{k}\lambda_i g_i\left(x^2,y^2\right)\right)$ are positive definite for i = 1, 2, …. ℓ

(ii) $\bar{p} \neq 0$ implies $\sum_{i=1}^{k} \lambda_i \nabla_y \left(\nabla_{yy} f_i \left(\bar{x}^1, \bar{y}^1 \right) \bar{p} \right) \bar{p} \neq 0$ and

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_y \left(\nabla_{yy} g_i \left(\bar{x}^2, \bar{y}^2 \right) \bar{p} \right) \bar{p} \neq 0$$

(iii) The vectors $\left\{ \nabla_{y^1} f_1 - \bar{z}_1^2 \ ,\ldots \nabla_{y^1} f_\ell - \bar{z}_\ell^{\,2} \right\}$ and $\left\{ \nabla_{y^2} g_\ell - \bar{z}_1^{\,2}, \ldots \nabla_{y^2} g_\ell - \bar{z}_\ell^{\,2} \right\}$ are linearly independent.

(iv) The vector

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_y \left(\nabla_{yy} f_i \left(\bar{x}^1, \bar{y}^1 \right) \bar{p} \right) \bar{p} \notin \text{span} \left\{ \nabla_y f_\ell \left(\bar{x}^1, \bar{y}^1 \right), \ldots, \nabla_y f_k \left(\bar{x}^1, \bar{y}^1 \right) \right\} \setminus \{0\}$$

and

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_y \left(\nabla_{yy} g_i \left(\bar{x}^2, \bar{y}^2 \right) \bar{p} \right) \bar{p} \notin \text{span} \left\{ \nabla_y g_1 \left(\bar{x}^2, \bar{y}^2 \right), \ldots, \nabla_y g_k \left(\bar{x}^2, \bar{y}^2 \right) \right\} \setminus \{0\}$$

If the generalized convexity hypothesis and conditions (i) to (iv) of weak duality theorem 3.1 are satisfied, then $\left(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{\lambda} \right)$ is an efficient solution for (MHD).

3.4 Special cases

(i) These results can be extended to higher order fractional programming problem.

(ii) These results can be extended to cone constraint conditions.

(iii) These results can be extended to generalized univex functions.

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