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Strong L^q Solutions of the Boussinesq System in R^m

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Abstract **— In this paper, we consider the m-dimensional Boussinesq equations, and show the time decay properties for the strong solutions to the Boussinesq system in the usual Sobolev space.**

Keywords **— Boussinesq equations; strong solutions; decay property.**

I. Introduction and Main Result

The aim of this paper, is to consider the following m-dimensional Boussinesq equations with the incompressibility condition:

$$\begin{cases}\partial_t u-\Delta u+(u\cdot\partial)u+\partial\Pi=\theta e_m, & (x,t)\in\mathbb{R}^m\times(0,\infty),\\ \partial_t\theta-\Delta\theta+(u\cdot\partial)\theta=0,\\ \partial\cdot u=0, \quad (\partial=\nabla=\text{grad})\\ u(x,0)=a, \quad \theta(x,0)=b,\end{cases} \tag{1}$$

where $u=u(x,t)$ and $\theta=\theta(x,t)$ denote the unknown velocity field and the scalar temperature, $\Pi=\Pi(x,t)$ is the scalar pressure, $\mu,\kappa>0$ are the constant kinematic viscosity and the thermal diffusivity respectively, $e_m=(0,0,\cdots,1)^T$. While a,b with $\nabla\cdot a=0$ in the sense of distribution are given initial data.

The problem of global existence or finite time blow-up of smooth solutions for 3D incompressible Euler or Navier-Stokes equations has been one of the most outstanding open questions in applied analysis, as well as that for 3D incompressible magneto hydrodynamics (MHD) equations. This challenging question has attracted significant attention. However, the main difficulty for this problem is to understand the effect of vortex stretching, which is absent in the 2D incompressible Euler or Navier-Stokes equations. To understand the vortex stretching effect for 3D flows, various simplified model equations have been proposed in the literature. The Boussinesq system is one of the most commonly used models, since it shares a similar vortex stretching effects in the 3D incompressible flows. The Boussinesq system has important roles in atmospheric sciences [1], and in many geophysical applications [2]. For these reasons, this system is studied systematically by scientists from different domains.

In recent years, there are a lot of literatures studying the existence of solutions for the system (1). More precisely, the blow up criteria are established, see [3,4,5,6], global well-posedness has been got in various function spaces and for different viscosities in [7,8,9,10,11,12] , [13,14,15,16,17] gave the time-decay properties. Strong solutions in different suitable function spaces are also been researched extensively. One may refer to [7,9,10,11] and the reference therein.

In this paper, we are interested in the time decay property of the weak solution of the system (1). Here and in what follows we denote by PL^p the subspace of $L^p(\mathbb{R}^m;\mathbb{R}^m)$ with the divergence condition $\nabla\cdot u=0$.

Our paper is inspired by the results in Tosio Kato [18], in which he investigated the time decay properties for the Navier-Stokes equations with u and ∂u. Based on the method in [18], we further investigate the time decay rates for the strong solutions to (1).

More precisely, our main results read as follows, where BC denotes the class of bounded and continuous functions.

Theorem 1.1 Suppose $a\in PL^m$, $b\in L^{\frac{m}{3}}$. Then there exists $T>0$ and a unique solution (u,θ) such that

$$\begin{aligned} t^{(1-\frac{m}{q})/2}u&\in BC([0,T);PL^q), \quad \text{for } m\le q\le\infty,\\ t^{\frac{3}{2}-\frac{m}{2q}}\theta&\in BC([0,T);L^q), \quad \text{for } \frac{2}{5}m<q\le\frac{1}{2}m,\\ t^{\frac{1}{2}}\partial u&\in BC([0,T);PL^m),\\ t^{2-\frac{m}{2q}}\partial\theta&\in BC([0,T);L^q), \quad \text{for } \frac{m}{3}<q\le\frac{m}{2}.\end{aligned}$$

Theorem 1.2 If there is $\lambda>0$ such that $\|a\|_m, \|b\|_{\frac{m}{3}}\le\lambda$, then the solution u,θ in Theorem 1.1 is global, that is, we could take $T=\infty$. In particular, $\|u(t)\|_q$ decays like $t^{-(1-\frac{m}{q})/2}$ as $t\to\infty$, including $q=\infty$, and $\|\theta(t)\|_q$ decays like $t^{-(\frac{3}{2}-\frac{m}{2q})}$, $\|\partial u(t)\|_m$ decays like $t^{-\frac{1}{2}}$, $\|\partial\theta(t)\|_q$ decays like $t^{-(2-\frac{m}{2q})}$, as $t\to\infty$.

In Theorem 1.1, when $m=3$, the results consist with [13]. Here we consider the all high dimensional Boussinesq equations.

This paper is structured as follows. In section 2, we introduce the corresponding integral equation and the lemma used later. In Section 3, we provide the proof of Theorem 1.1.

II. The Integral Equation and the Lemma

We rewrite Boussinesq equations (1.1) in the abstract form

$$\begin{cases}\partial_t u+Au+F(u,u)+K(\theta)=0,\\ \partial_t\theta+A\theta+F(u,\theta)=0,\end{cases} \tag{2}$$

where $A=-P\Delta=-\Delta P$, $K(\theta)=\tilde{P}\theta$, $P=-\nabla\Delta^{-1}\text{div}$, $\tilde{P}=\nabla\Delta^{-1}\partial_m$ and

$$F(u,v)=P(u\cdot\partial)v.$$

It is well known that P and $\tilde{P}$ are both the orthogonal projections of L^2 onto the subspace PL^2, and $P,\tilde{P}$ are extended to the bounded operators on L^p to PL^p, $1<p<\infty$. Due to A is essentially equal to $-\Delta$, e^{-tA} is essentially the heat operator, and $P,\tilde{P}$ commute with the Laplacian operator Δ. (2) is converted into the integral equation

$$\begin{cases} u = u_0 + Gu + H\theta, \\ \theta = \theta_0 + R(u,\theta), \end{cases} \tag{3}$$

where

$$u_0(t) = e^{-tA}a,$$

$$Gu(t) = -\int_0^t e^{-(t-s)A} F(u(s), u(s))ds, \tag{4}$$

$$H\theta(t) = \int_0^t e^{-(t-s)A} K(\theta(s))ds,$$

$$\theta_0(t) = e^{-tA}b,$$

$$R(u,\theta)(t) = -\int_0^t e^{(t-s)A} F(u,\theta)ds. \tag{5}$$

Lemma 2.1 Let $1 < p, q < \infty$. The following statements hold:

$$\| e^{-tA}u \|_q \le Ct^{-(\frac{m}{p}-\frac{m}{q})/2} \| u \|_p, \quad 1 < p \le q, \tag{6}$$

$$\| \partial e^{-tA}u \|_q \le Ct^{-(1+\frac{m}{p}-\frac{m}{q})/2} \| u \|_p, \tag{7}$$

$$\| F(u,v) \|_p \le C \| u \|_r \| \partial v \|_s, \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s}. \tag{8}$$

Here C denotes various constants that do not depend on the individual functions u, v. (6) bases on the fact that the $L^p(\mathbb{R}^m)$ -norm of the heat kernel is proportional to $t^{-(m-\frac{m}{p})/2}$, $A = -P\Delta$ is essentially identical with $-\Delta$. (7) bases on the property of the derivatives of the heat kernel, (8) is simply the Hölder inequality.

Combining (6)-(7) with (8), with a slight change of notation, we get

$$\| Gu(t) \|_{\frac{m}{\gamma}} \le C \int_0^t (t-s)^{-(\alpha+\beta-\gamma)/2} \| u(s) \|_{\frac{m}{\alpha}} \| \partial u(s) \|_{\frac{m}{\beta}} ds, \tag{9}$$

$$\| \partial Gu(t) \|_{\frac{m}{\gamma}} \le C \int_0^t (t-s)^{-(1+\alpha+\beta-\gamma)/2} \| u(s) \|_{\frac{m}{\alpha}} \| \partial u(s) \|_{\frac{m}{\beta}} ds, \tag{10}$$

where $\alpha, \beta, \gamma > 0$, $\gamma \le \alpha + \beta < m$. Now we show a result about the convergence of a defect integral.

Lemma 2.2 Set $I = \int_0^t (t-s)^{-\frac{1}{2}} s^{-1+\frac{\delta}{2}} ds$, with $0 < \delta < 1$. Then I is convergent, and can be expressed as $Ct^{-\frac{1-\delta}{2}}$.

Proof: The convergence and the new expression can be proved by a simple computation with a change of variable. We set $s = ts'$, thus

$$\begin{aligned} I &= \int_0^t (t-s)^{-\frac{1}{2}} s^{-1+\frac{\delta}{2}} ds \\ &= \int_0^1 (t-ts')^{-\frac{1}{2}} (ts')^{-1+\frac{\delta}{2}} t ds' \\ &= t^{-\frac{1}{2}} t^{-1+\frac{\delta}{2}} t \int_0^1 (1-s)^{-\frac{1}{2}} s^{-1+\frac{\delta}{2}} ds \\ &= Ct^{-\frac{1-\delta}{2}}, \end{aligned}$$

where by $0 < \frac{1}{2} < 1$, $0 < 1 - \frac{\delta}{2} < 1$, the defect integral $\int_0^1 (1-s)^{-\frac{1}{2}} s^{-1+\frac{\delta}{2}} ds$ is convergent.

III. The Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Firstly, we focus on solve (3) by successive approximation, with u_0, θ_0 given by (4), (5) respectively and construct

$$u_{n+1} = u_0 + Gu_n + H\theta_{n+1}, \quad n = 0,1,2,\cdots, \tag{11}$$

$$\theta_{n+1} = \theta_0 + R(u_n, \theta_n), \quad n = 0,1,2,\cdots. \tag{12}$$

We will prove that the u_n and θ_n exist by induction, and satisfy the estimates as follows:

$$t^{\frac{1-\delta}{2}} u_n \in BC([0,\infty); PL^{\frac{m}{\delta}}) \quad \text{with norm} \le K_n, \tag{13}$$

$$t^{\frac{1}{2}-\frac{\delta}{4}} \theta_n \in BC\left([0,\infty); L^{\frac{m}{\eta}}\right) \quad \eta = 2 + \frac{\delta}{2}, \quad \text{with norm} \le Q_n, \tag{14}$$

$$t^{\frac{1}{2}} \partial u_n \in BC([0,\infty); PL^m) \quad \text{with norm} \le K'_n, \tag{15}$$

$$t^{2-\frac{m}{2f}} \partial\theta_n \in BC([0,\infty); L^f) \quad \text{with norm} \le Q'_n. \tag{16}$$

Here δ is any fixed number, $0 < \delta < 1$. Moreover, all the functions (13-16) vanish at $t = 0$.

For $n = 0$, taking $q = \frac{m}{\delta}$ and $q = m$ in (6) and (7) respectively, and $p = m$, $u = a$, with the definition in (4), we obtain

$$\|u_0(t)\|_{\frac{m}{\delta}} \le ct^{-(1-\delta)/2} \|a\|_m, \tag{17}$$

$$\|\partial u_0(t)\|_m \le ct^{-1/2} \|a\|_m. \tag{18}$$

Taking $q = \frac{m}{\eta}$, $p = \frac{m}{3}$, and $q = f$, $u = b$ in (6) and (7) respectively, with the definition in (5), we get

$$\|\theta_0(t)\|_{\frac{m}{\eta}} \le ct^{-\frac{3-\eta}{2}} \|b\|_{\frac{m}{3}} \tag{19}$$

$$\|\partial\theta_0(t)\|_f \le ct^{-(-4-\frac{m}{f})/2} \|b\|_{\frac{m}{3}} \tag{20}$$

Assuming now that (13)-(16) are true for n, we proceed to prove them for n+1. For first term $u_0(t)$ and $\theta_0(t)$ on the right side of (11-12) have been estimated in (17-20), only if we just take

$$K_0 = K'_0 = c\|a\|_m, Q_0 = Q'_0 = c\|b\|_{\frac{m}{3}} \tag{21}$$

We now focus on the terms Gu_n, $H\theta_{n+1}$ and $R(\theta_n, u_n)$.

The $L^{\frac{m}{\delta}}$ -norm of the term Gu_n is estimated by taking $\gamma = \alpha = \delta, \beta = 1$ in (9), thus due to Lemma 2, we have

$$\|Gu_n\|_{\frac{m}{\delta}} \le C\int_0^t (t-s)^{-\frac{1}{2}} \|u_n(s)\|_{\frac{m}{\delta}} \|\partial u_n(s)\|_m ds$$

$$\|u_n\|_{\frac{m}{\delta}} \le ct^{-\frac{1-\delta}{2}} K_n, \|\partial u_n\|_m \le ct^{-\frac{1}{2}} K'_n.$$

In order to estimate the $L^{\frac{m}{\delta}}$ -norm of $H\theta_{n+1}$, we set $q = \frac{m}{\delta}, p = \frac{m}{\eta}$ in (6), then we get

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$$\|H\theta_{n+1}\|_{\frac{m}{\delta}} \le \int_0^t \left\| e^{-(t-s)A}\tilde{I}\,\theta_{n+1}\right\|_{\frac{m}{\delta}} ds$$
$$\le C\int_0^t (t-s)^{-\frac{\eta-\delta}{2}} \|\theta_{n+1}\|_{\frac{m}{\eta}} ds. \qquad (23)$$

Now we turn to estimate θ_{n+1} in (12), that is the term $R(\theta_n, u_n)$. According to (6) and (8), we have

$$\|R(u_n,\theta_n)\|_{\frac{m}{\eta}} \le \int_0^t \left\| e^{-(t-s)A} F(u,\theta)\right\|_{\frac{m}{\eta}} ds$$
$$\le C\int_0^t (t-s)^{-(\frac{m}{\sigma}-\eta)/2} \|u_n\cdot\partial\theta_n\|_{\sigma} ds$$
$$\le C\int_0^t (t-s)^{-(\frac{m}{\sigma}-\eta)/2} \|u_n\|_{\frac{m}{\delta}} \|\partial\theta_n\|_f\, ds,$$

where $\frac{1}{\sigma}=\frac{\delta}{m}+\frac{1}{f}$. The assumption

$$\|u_n\|_{\frac{m}{\delta}} \le ct^{-\frac{1-\delta}{2}} K_n, \|\partial\theta_n\|_f \le ct^{-(4-\frac{m}{f})/2} Q'_n$$

implies

$$\|R(u_n,\theta_n)\|_{\frac{m}{\eta}} \le \int_0^t (t-s)^{-(\frac{m}{\sigma}-\eta)/2} s^{-\frac{1-\delta}{2}} K_n s^{-(4-\frac{m}{f})/2} Q'_n ds$$
$$\le \int_0^t (t-s)^{-(\frac{m}{\sigma}-\eta)/2} s^{-\frac{3}{2}+\frac{\delta}{2}+\frac{m}{2f}} ds K_n Q'_n$$
$$\le t^{-\frac{3-\eta}{2}} \int_0^1 (1-s)^{-(\frac{m}{\sigma}-\eta)/2} s^{-\frac{3}{2}+\frac{\delta}{2}+\frac{m}{2f}} ds K_n Q'_n, \qquad (24)$$

where we used the similar method in Lemma 2.2, and to guarantee the convergence of the defect integral above, we should require

$$0<\frac{m}{\sigma}-\eta<2 \text{ that is } 0<\delta+\frac{m}{f}-\eta<2 \qquad (25)$$

and

$$0<-\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2f}<1 \quad \text{that is} \quad 0<1+\delta+\frac{m}{f}<3\ . \qquad (26)$$

Complying (12), (19) with (24), we have

$$\|\theta_{n+1}\|_{\frac{m}{\eta}} \le \|\theta_0\|_{\frac{m}{\eta}} + \|R(u_n,\theta_n)\|_{\frac{m}{\eta}}$$
$$\le t^{-\frac{3-\eta}{2}} Q_0 + t^{-\frac{3-\eta}{2}} K_n Q'_n$$
$$= t^{-\frac{3-\eta}{2}} Q_{n+1}. \qquad (27)$$

Here we set $Q_{n+1}=Q_0+K'_n Q'_n$.

Substituting (27) into (23), we obtain

$$\|H\theta_{n+1}\|_{\frac{m}{\delta}} \le C\int_0^t (t-s)^{-\frac{\eta-\delta}{2}} s^{-\frac{3-\eta}{2}} ds Q_{n+1}$$
$$\le Ct^{-\frac{\eta-\delta}{2}} t^{-\frac{3-\eta}{2}} t\int_0^1 (1-s)^{-\frac{\eta-\delta}{2}} s^{-\frac{3-\eta}{2}} ds Q_{n+1}$$
$$\le Ct^{-\frac{1-\delta}{2}} \int_0^1 (1-s)^{-\frac{\eta-\delta}{2}} s^{-\frac{3-\eta}{2}} ds Q_{n+1}$$
$$\le Ct^{-\frac{1-\delta}{2}} Q_{n+1}. \qquad (28)$$

where we used the method in Lemma 2.2.

Finally, according to (11), (17), (22) and (28), we get

$$\|u_{n+1}\|_{\frac{m}{\delta}} \le \|u_0\|_{\frac{m}{\delta}} + \|Gu_n\|_{\frac{m}{\delta}} + \|H\theta_{m+1}\|_{\frac{m}{\delta}}$$
$$\le Ct^{-\frac{1-\delta}{2}} K_0 + Ct^{-\frac{1-\delta}{2}} K_n K'_n + Ct^{-\frac{1-\delta}{2}} Q_{n+1}$$
$$\le Ct^{-\frac{1-\delta}{2}} K_{n+1}. \qquad (29)$$

Here we set $K_{n+1}=K_0+K_nK'_n+Q_{n+1}$.

Thus (27) and (29) by iteration showed the correction of (14) and (13) respectively. Similarly, we deal with (15) and (16). Differentiating (11) and (12), we get ∂u_{n+1} and $\partial\theta_{n+1}$ as follow

$$\partial u_{n+1}=\partial u_0+\partial Gu_n+\partial H\theta_{n+1} \qquad (30)$$
$$\partial\theta_{n+1}=\partial\theta_0+\partial R(u_n,\theta_n). \qquad (31)$$

Applying (10) and the assumption (13), (15) are true for n, we have

$$\|\partial Gu_n\|_m \le C\int_0^t (t-s)^{-\frac{1+\delta}{2}} \|u_n(s)\|_{\frac{m}{\delta}} \|\partial u_n(s)\|_m ds$$
$$\le C\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{-1+\frac{\delta}{2}} ds K_n K'_n$$
$$\le Ct^{-\frac{1+\delta}{2}} t^{-1+\frac{\delta}{2}} t\int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{-1+\frac{\delta}{2}} ds K_n K'_n$$
$$\le Ct^{-\frac{1}{2}} K_n K'_n, \qquad (32)$$

where we noticed the convergence of the defect integral $\int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{-1+\frac{\delta}{2}} ds$. By (10) again, we obtain

$$\|\partial H\theta_{n+1}\|_m \le \int_0^t \left\| e^{-(t-s)A}\tilde{I}\,\theta_{n+1}\right\|_m ds$$
$$\le C\int_0^t (t-s)^{-(\frac{m}{f}-1)/2} \|\partial\theta_{n+1}\|_f ds. \qquad (33)$$

Thus, we need to estimate $\|\partial\theta_{n+1}\|_f$, we only need to do the term $\|\partial R(u_n,\theta_n)\|_f$. Combining (10), (8) with the assumption (3.3), (3.6) are true for n, we get

$$\|\partial R(u_n,\theta_n)\|_f \le \int_0^t \left\|\partial e^{-(t-s)A} F(u_n,\theta_n)\right\|_f ds$$
$$\le C\int_0^t (t-s)^{-(1+\frac{m}{k}-\frac{m}{f})/2} \|F(u_n,\theta_n)\|_f ds$$
$$\le C\int_0^t (t-s)^{-(1+\frac{m}{k}-\frac{m}{f})/2} \|u_n(s)\|_{\frac{m}{\delta}} \|\partial\theta_n(s)\|_f ds$$
$$\le C\int_0^t (t-s)^{-(1+\frac{m}{k}-\frac{m}{f})/2} s^{-\frac{1-\delta}{2}} s^{-(4-\frac{m}{f})/2} K_n Q'_n ds,$$

where $\frac{1}{k}=\frac{\delta}{m}+\frac{1}{f}$, thus by similar method in Lemma 2, we have

$$\|\partial R(u_n,\theta_n)\|_f \le C\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{-\frac{5-\delta-\frac{m}{f}}{2}} ds K_n Q'_n$$
$$= Ct^{-\frac{1+\delta}{2}} t^{-\frac{5-\delta-\frac{m}{f}}{2}} t\int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{-\frac{5-\delta-\frac{m}{f}}{2}} ds K_n Q'_n$$
$$= Ct^{-2+\frac{m}{2f}}\int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{-\frac{5-\delta-\frac{m}{f}}{2}} ds K_n Q'_n .$$

Here to keep the convergence of the defect integral above, we should ask for

$$0<\frac{5-\delta-\frac{m}{f}}{2}<1 \text{ that is } -1<\frac{m}{f}+\delta<5. \quad (34)$$

Therefore

$$\|\partial R(u_n,\theta_n)\|_f \le Ct^{-2+\frac{m}{2f}} K_n Q'_n . \quad (35)$$

Substituting (20) and (35) into the formula (31), we obtain

$$\|\partial\theta_{n+1}\|_f \le Ct^{-2+\frac{m}{2f}}(Q'_0+K_nQ'_n). \quad (36)$$

Here we could set $Q'_{n+1}=Q'_0+K_nQ'_n$. Putting (36) into (33), we get

$$\|\partial H\theta_{n+1}\|_m \le \int_0^t (t-s)^{-(\frac{m}{f}-1)/2} s^{-2+\frac{m}{2f}} ds Q'_{n+1}$$
$$\le Ct^{-(\frac{m}{f}-1)/2} t^{-2+\frac{m}{2f}} t\int_0^1 (1-s)^{-(\frac{m}{f}-1)/2} s^{-2+\frac{m}{2f}} ds Q'_{n+1}$$
$$= Ct^{-\frac{1}{2}} Q'_{n+1}, \quad (37)$$

to keep the defect integral is convergent, we should demand

$$0<(\frac{m}{f}-1)/2<1 \quad \text{that is} \quad \frac{1}{2}<\frac{m}{f}<3 \quad (38)$$

And

$$0<\frac{1}{2}+\frac{m}{2g}-\frac{m}{2f}<1 \quad \text{that is} \quad -1<\frac{m}{f}-\frac{m}{g}<1 \quad (39)$$

Finally, bonding (18), (32), (37) with (30), we have

$$\|\partial u_{n+1}\|_m \le Ct^{-\frac{1}{2}}(K'_0+K_nK'_n+Q'_{n+1})$$
$$\le Ct^{-\frac{1}{2}} K'_{n+1},$$

here, we choose $K'_{n+1}=K'_0+K_nK'_n+Q'_{n+1}$.

These computations also justify the continuity at $t=0$, with values zero, of the functions (13)-(16) with subscript n+1. Indeed, due to this is true for subscript n by induction hypothesis, the constants K_n, K'_n, Q_n, Q'_n can be made arbitrarily small if we restrict ourselves to a small time interval $[0,\tau)$.

Since the same is true for K_0, K'_0, Q_0, Q'_0 as shown above, it follows the system of recurrence inequalities K_n, K'_n, Q_n, Q'_n that $K_{n+1}, K'_{n+1}, Q_{n+1}, Q'_{n+1}$ are also arbitrarily small when we restricted to $[0,\tau)$.

The system of recurrence inequalities K_n, K'_n, Q_n, Q'_n can be solved in the same way as in [19]. It turns out that there is a number $\lambda>0$, such that if $K_0, K'_0, Q_0, Q'_0 \le \lambda$, then K_n, K'_n, Q_n, Q'_n are bounded by a fixed constant K . In view of (21), this is true if $\|a\|_m$ and $\|b\|_{\frac{m}{3}}$ are sufficiently small. Thus, the sequences (13)-(16) are uniformly bounded, and their uniform convergence on $(0,\infty)$ can be showed as in [19].

Finally, for the case $q=\infty$ in Theorem 1.1 can be dealt with by the Gagliardo-Nirenberg inequality $\|u(t)\|_\infty^2 \le C\|u(t)\|_{2m}\|\partial u(t)\|_{2m}$. This also finishes the proof of Theorem 1.2.

Remark 3.1 Through the paper, to keep the convergence of the defect integral, we require (25), (26), (34), (38) and (39). It is easy to see that (25), (26) and (34) imply that $2+\frac{\delta}{2}<\frac{m}{f}+\delta<3$; (38) and (39) imply that $2<\frac{m}{f}<3$. And the intersection of the two inequalities is non-empty.

V. Conclusion

After converting the Boussinesq equation into the integral form and solving it by successive approximation, we get the unique solution (u,θ) in the function spaces $L^p(R^m)$ with the proper p in Theorem 1.1. Also, we get the decay estimate as $t\to\infty$ in Theorem 1.2.

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