Degree Polynomial of Some Special Classes of Trees

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I. INTRODUCTION

By a graph G = (V, E), we mean a finite, undirected simple graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by F. Harary [?]. In a graph G = (V, E), the degree of a vertex v is defined to be the number of edges incident with v and is denoted by deg(v). The minimum of \{\deg(v) : v \in V(G)\} is denoted by \(\delta(G)\) and the maximum of \{\deg(v) : v \in V(G)\} is denoted by \(\Delta(G)\). A vertex u in G is called an isolated vertex if \(\deg(u) = 0\) and is called a pendant vertex if \(\deg(u) = 1\). For appositive integer i, \(\deg(G,i) = \{v \in V(G) : \deg(v) = i\}\) and \(\deg(G,i) = \{\deg(G,i)\}\) are the vertex subset of V (G) of degrees equal to i and the number of vertices of G of degrees equal to i, respectively. The open neighbourhood of a vertex \(u \in V(G)\) is \(N(u) = \{v \in V(G) : uv \in E(G)\}\) and the closed neighbourhood is \(N[u] = N[v] \cup \{v\}\).

A tree is a graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree. All graphs in this paper are assumed to be tree.

There are many ways to characterized the graph. A graph can be characterized by a drawing, by a number, by a matrix or by a polynomial. The characterization of graphs by a single topological index is usually impossible. For example, it is possible to find infinite pairs of graphs with the same Wiener index. It is an open question to find a topological index characterizing graphs.

On the other hand, it is possible to characterize graphs by matrices. A well-known example of such matrices is an adjacency matrix. But the characterization of graphs by polynomials is a new branch of research in modern graph theory. One of the algebraic graph theory branches is graph polynomials and there are many graph polynomials that have been introduced and studied widely. One of the most general approaches to graph polynomials was proposed by Farrell [?], in his theory of F-polynomials of a graph. According to Farrell, any such polynomial corresponds to a strictly prescribed family of connected subgraphs of the respective graph. For the matching polynomial of a graph [8], this family consists of all the edges of, for the independence polynomial of [?], this family consists of all the stable (independent) sets of, for the chromatic polynomial of [3], this family consists of all the color classes of, for the domination polynomial of [1], this family consists of all the domination sets of and for the resolving polynomial of, this family includes all the resolving sets of . In fact, various aspects of combinatorial information concerning a graph is stored in the coefficients of a specific graph polynomial.

The present author and Soner N. D have been introduced the degree polynomial of a graph[12] and defined it as \(\deg(G,x) = \sum_{i=0}^{\Delta} \deg(G,i)x^i\), where \(\deg(G,i)\) is the number of vertices in G of degree i, for \(\delta(G) \leq i \leq \Delta(G)\). The roots set of the polynomial \(\deg(G,x)\) is denoted by \(\mathcal{Z}(\deg(G,x))\) and is called the degree roots of a graph G. They have been obtained some properties of the degree polynomial of a graph. Exact formulae of degree polynomial of some standard graphs and binary graph operations, aside from union, join and corona product are obtained. They also, defined the graphical degree polynomial. The necessary and sufficient conditions for any polynomial \(P(x)\) to be graphical degree polynomial is established.

In this paper, the degree polynomial formula of some special classes of trees as star \(k_{1, n-1}\), bistar (double star) \(S_{n,m}\), broum \(B_{n,k}\), double broom \(D(a, r, b)\) and some classes of thorn trees as caterpillar T \((b_1, b_2, \ldots, b_n)\), thorn rods \(P_{b,t}\) and thorn stars \(S_{h, t}\) are presented. Finally, the general degree polynomial formula of a Kragujevac (Gutman) trees T \((\beta_1, \beta_2, \ldots, \beta_n)\) for \(d \geq 2\) is computed.

Definition 1.1. A star \(K_{1, n+1}\) is a tree with one internal (center) vertex and n-1 pendant vertices (leaves).

Theorem 1.2. The degree polynomial of a star \(K_{1, n+1}\) with at least n-2 vertices is 
\[\deg(K_{1, n+1}, x) = (n-1)x + an^{n-1}\]

Proof. For the star tree \(K_{1, n+1}\) with n \geq 2 vertices, \(\delta(K_{1, n+1}) = 1\) and \(\Delta(K_{1, n+1}) = n-1\), let \(v_0, v_1, v_2, \ldots, v_{n-1}\) be the vertex set of \(K_{1, n+1}\), where \(v_0\) be the center vertex and \(v_1, v_2, \ldots, v_{n-1}\) be the
Theorem 1.4. The degree polynomial of a bistar graph $S_{m,n}$ with at least $n \geq 2$ vertices is

$$\text{Deg}(S_{m,n},x) = (m+n)x + x^m + x^n.$$ 

Proof. Let $\{u_0,u_1,\ldots,u_m,u_{m+1},u_{m+2},\ldots,u_n\}$ be the set vertex of the bistar graph $S_{m,n}$ with $m+n+2$ vertices, where $u_0$ and $u_0$ be the central vertices. Suppose, without lose of generality, that $m \leq n$. Then $\Delta(S_{m,n}) = 1$ and $\Delta(S_{m,n}) = n$ and

$$\text{Deg}(S_{m,n},1) = \{u_1,u_m,u_{m+1},u_{m+2},\ldots,u_n\}.$$ 

Hence, $\text{deg}(S_{m,n,1}) = m + n, \text{deg}(S_{m,n,m}) = \text{deg}(S_{m,n,n}) = 1$, and $\text{deg}(S_{m,n,i}) = 0$, for every $i \notin \{1,m,n\}$

Therefore,

$$\text{Deg}(S_{m,n},x) = \sum_{i=1}^{n} \text{deg}(S_{m,n,i})x^i = (m+n)x + x^m + x^n$$

Corollary 1.5. for $n \geq 2$

Definition 1.6. [10, 11] The broom graph $B_{n,d}$ is a graph on $n+d$ vertices, obtained by attaching $d \geq 2$ pendant vertices to one extreme vertex of the path $P_n$, for $n \geq 2$.

Theorem 1.7. The degree polynomial of a broom graph $B_{n,d}$ with at least $n + d$ vertices is

$$\text{Deg}(B_{n,d},x) = (d+1)x + (n-2)x^2 + x^{d+1}.$$ 

Proof. Let $B_{n,d}$ be the broom tree of order $n + d$, for $n, d \geq 2$, which obtained by attaching $d$ vertices (namely, $u_1, \ldots, u_d$) to the extreme vertex $u_n$ of a path $P_n$ with vertex set $u_1, u_2, \ldots, u_n$.

Then $\Delta(B_{n,d}) = 1$, $\Delta(B_{n,d}) = d + 1$, and $\text{Deg}(B_{n,d},1) = \{u_1, u_1, u_2, \ldots, u_d\}$

Therefore,

$$\text{Deg}(B_{n,d},x) = \sum_{i=1}^{d+1} \text{deg}(B_{n,d,i})x^i = (d+1)x + (n-2)x^2 + x^{d+1}$$

Definition 1.8. [14] The double broom $D(a, n, b)$ is a graph on $a + n + b$ vertices, for $n \geq 2$, obtained by attaching a $\geq 1$ pendant vertices to one extreme of the path $P_n$ and $b \geq 1$ pendant vertices to the other.

Theorem 1.9. The degree polynomial of a double broom graph $D(a, n, b)$ with at least $a+n+b$ vertices is

$$\text{Deg}(D(a,n,b),x) = (a+b)x + (n-2)x^2 + x^{a+b+1}$$ 

Proof. For positive integer numbers $a \geq b \geq 1$ and $n \geq 2$, let $u_1, u_2, \ldots, u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+b}$ be the vertex set of the double broom graph $D(a,n,b)$, which obtained from a path.
P_n of vertices u_1, w_2, ..., w_n by attaching the pendant vertices u_1, u_2, ..., u_k to the extreme vertex w of P_n and attaching the pendant vertices u_1, u_2, ..., u_k to the other extreme vertex w of P_n. Then
\[ \text{Deg}(D(a, n, b), 1) = \{v_1, v_2, ..., v_{n-1}\}, \]
\[ \text{Deg}(D(a, n, b), 2) = \{w_1, w_2, ..., w_{n-1}\}, \]
and \( \text{Deg}(D(a, n, b), i) = \phi \), for every \( 1 \leq i \leq a \) and \( i \in \{1, 2, b+1, a+1\} \).

Thus, \( \text{deg}(D(a, n, b), 1) = a+b \), \( \text{deg}(D(a, n, b), 2) = n-2 \), \( \text{deg}(D(a, n, b), a+1) = \text{deg}(D(a, n, b), b+1) = 1 \) and \( \text{deg}(D(a, n, b), i) = 0 \), for every \( 1 \leq i \leq a \) and \( i \in \{1, 2, b+1, a+1\} \).

Therefore, \( \text{Deg}(D(a, n, b), x) = (a+b)x + (n-2)x^2 + x^{a+1} + x^{b+1} \)

**Corollary 1.10.** For positive integer numbers \( a \geq 1 \) and \( n \geq 2 \),
\[ \text{Deg}(D(a, n, b), x) = 2ax + (n-2)x^2 + x^{a+1} \]

**Definition 1.11.** A Spider \( Sp(n_1, n_2, ..., n_k) \), \( k \geq 2 \), is a tree with at most one vertex of degree more than two, called the center of Spider (if no vertex of degree more than two, then any vertex can be the center). A leg of a Spider is a path \( P_i \) with \( n_i \geq 1 \) vertices, \( 1 \leq i \leq k \), from the center to a vertex of degree one. Thus, a star with \( k \) legs is a Spider \( Sp(1, 1, ..., 1) \) of \( k \) legs, each of length 1, [4].

*Fig. 5. A spider tree \( Sp(n_1, n_2, ..., n_k) \).*

**Theorem 1.12.** The degree polynomial of a spider tree \( Sp(n_1, n_2, ..., n_k) \), \( k \geq 2 \) is
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), x) = kx + \left( \sum_{i=1}^{k} n_i - 1 \right)x^2 + x^k \]

**Proof.** Let \( Sp(n_1, n_2, ..., n_k) \) be a spider tree with central vertex \( v_0 \) and \( P_1, P_2, ..., P_k \), for \( k \geq 2 \) legs of lengths \( n_1, n_2, ..., n_k \), respectively and let \( V(P_i) = \{v_{i1}, v_{i2}, ..., v_{i{n_i}}\} \) be the set vertex of the leg \( P_i \), \( 1 \leq i \leq k \), where \( v_{i1} \) is the vertex that adjacent to the central vertex \( v_0 \) and \( v_{in_i} \) is the extreme vertex. Then
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), 1) = \{v_{i1} : 1 \leq i \leq k\} \]
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), 2) = \{v_{i1}, v_{i2}, ..., v_{i(n_i)-1} : 1 \leq i \leq k\} \]
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), k) = \{v_{ik} \} \]
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), 2) = k \left( \sum_{i=1}^{k} n_i - 1 \right) \]
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), k) = 1 \]

Therefore, \( \text{Deg}(Sp(n_1, n_2, ..., n_k), x) = kx + \left( \sum_{i=1}^{k} n_i - 1 \right)x^2 + x^k \)

**Corollary 1.13.** If \( n_1 = n_2 = ... = n_k = n \), for \( n \geq 2 \), then
\[ \text{Deg}(Sp(n_1, n_2, ..., n_k), x) = kx + (n(k-1)x^2 + x^k \]

**Definition 1.14.** For a positive integers \( r, s \) and \( w \), a wounded spider \( Sp(r, s) \) is a spider with \( r \geq 1 \) legs of length 1 and \( s \geq 1 \) legs of length 2.

**Corollary 1.15.** The degree polynomial of a wounded spider tree \( Sp(r, s) \), \( r, s \geq 1 \) with vertices \( r+s+1 \) is
\[ \text{Deg}(Sp(r, s), x) = (r+s)x + s^2x^2 + x^r+s \]

**Proof.** Let \( r \geq 1 \), let \( v_0 \) be the head vertex of a wounded spider tree \( Sp(r, s) \) with \( r \) legs of lengths one and \( s \) legs with lengths two. Let \( v_1, v_2, ..., v_r \) be the vertex set of legs with length one and \( u_1, u_2, ..., u_s \) be the vertex set of legs with length two, where \( w_1, w_2, ..., w_s \) be the pendant vertices. Then \( \text{deg}(Sp(r, s)) = 1, \text{Deg}(Sp(r, s)) = r+s \),
\[ \text{Deg}(Sp(r, s), 1) = \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_s\} \]
\[ \text{Deg}(Sp(r, s), 2) = \{v_1, v_2, ..., u_s\} \]
\[ \text{Deg}(Sp(r, s), r+s) = \{v_0\} \]
and \( \text{Deg}(Sp(r, s), i) = \phi \), for every \( 1 < i < r+s \) and \( i \neq 2 \).

Therefore, \( \text{Deg}(Sp(r, s), x) = (r+s)x + s^2x^2 + x^{r+s} \)

**Definition 1.16.** An olive tree \( T_k \), is a rooted tree consisting of \( k \geq 2 \) branches, the \( i-th \) branch is a path with a length \( i \geq 1 \). In other words, \( T_k \) is a spider \( Sp(1, 2, 3, ..., k) \).

**Corollary 1.17.** The degree polynomial of an olive tree \( T_k \), \( k \geq 2 \) is
\[ \text{Deg}(T_k, x) = kx + \left( \frac{k}{2} \right)x^2 + x^k \]

**Proof.** Let \( T_k \) for \( k \geq 2 \) be an olive tree with central vertex \( v_0 \) and let \( P_1, P_2, ..., P_k \) be the legs of \( T_k \) of lengths \( 1, 2, ..., k \), respectively and let \( V(P_i) = \{v_{i1}, v_{i2}, ..., v_{ik}\} \) be the set vertex of the leg \( P_i \), \( 1 \leq i \leq k \). Then \( \text{deg}(T_k, 1) = \{v_{i1}, v_{i2}, ..., v_{ik}\} \)
\[ \text{Deg}(T_k, 2) = \{v_{ik} : 1 \leq i, j \leq k \text{ and } i \neq j\} \]
\[ \text{Deg}(T_k, k) = \{v_0\} \]
\[ \text{Deg}(T_k, j) = \phi \text{, for every } 3 \leq j \leq k-1. \]
Thus, 
\[ \text{deg}(T_k, 1) = k, \text{ deg}(T_k, 2) = \sum_{i=1}^{k-1} (\binom{k}{2}). \]
\[ \text{deg}(T_k, k) = 1 \text{ and } \text{deg}(T_k, j) = 0, \text{ for every } 3 \leq j \leq k-1. \]
Therefore, \( \text{Deg}(T_k, x) = kx + k. \)

**Definition 1.18.** A binary tree \( T \) is a rooted tree where each vertex has degree at most three. The depth of a vertex \( v \) in a tree \( T \) is the length of the path from the root \( v \) to the vertex \( v \), the root \( v \) itself has depth zero. A vertex with degree one is called a leaf of the tree, all non-leaves are internal vertices. The height \( h \) of a binary tree is the length of the longest path from the root to any vertex. 

A **full binary tree** \( T_b \) is a binary tree where all internal vertices (except the root) have degree equals to three and all leaves are at the same depth \( h \).

![Fig. 6. A full binary tree \( T_b \).](image)

**Theorem 1.19.** For a full binary tree \( T_b \) of height \( h \geq 2 \) the degree polynomial of is of the form 2 the degree polynomial is of the form
\[ \text{Deg}(T_b, x) = 2^h x + x^2 + (2^h - 2) x^3. \]
**Proof.** Let \( T_b \), for \( h \geq 2 \), be a full binary tree of height \( h \), let \( v_b \) be the root vertex of \( T_b \) (the vertex of level \( h = 0 \)) and let \( \nu_{b1}, \nu_{b2}, \ldots, \nu_{b2^h} \), for \( 1 \leq i \leq b \), be the vertex set of a level \( i \) of \( T_b \). Then \( \Delta(T_b) = 1 \), \( \Delta(T_b) = 3 \) and \[ \text{Deg}(T_b, 1) = \nu_{b1}, \text{ Deg}(T_b, 2) = \nu_{b2}, \text{ Deg}(T_b, 3) = \nu_{b1}, \nu_{b2}, \ldots, \nu_{b2^h}. \]
Thus \[ \text{deg}(T_b, 1) = 2^h, \text{ deg}(T_b, 2) = 1, \text{ and } \text{deg}(T_b, 3) = \sum_{i=1}^{h-1} 2^i = 2^h - 2. \]
Therefore, \( \text{Deg}(T_b, x) = 3 \sum_{i=1}^{h-1} \text{deg}(T_b, i)x^i = 2^h x + x^2 + (2^h - 2)x^3. \)

**Definition 1.20.** [7] The thorn (thorny) graph \( G^* \) is a graph obtained from a parent \( n \)-vertex graph \( G \) by adjoining \( p \geq 1 \) new pendant vertices to the \( i \) th vertex of \( G \), for \( i = 1, 2, \ldots, n. \)

If the parent graph \( G \) is a tree, then we speak of thorn trees [2]. The concept of thorn graphs was introduced by I. Gutman [7].

**Definition 1.21.** [13] A caterpillar tree \( T(b_1, b_2, \ldots, b_n) \) is a thorn tree obtained from a path \( P_{n+2} \) on \( n + 2 \) vertices labelled consecutively as \( u_1, u_2, \ldots, u_n, u_{n+1}, u_{n+2} \) by attaching \( b_i \) pendant vertices to the vertex \( u_{i+1} \), for \( i = 1, 2, \ldots, n. \)

![Fig. 7. A caterpillar tree \( T(b_1, b_2, \ldots, b_n) \).](image)

**Theorem 1.22.** For a caterpillar tree \( T(b_1, b_2, \ldots, b_n) \), the degree polynomial is of the form
\[ \text{Deg}(T(b_1, b_2, \ldots, b_n), x) = \left( \sum_{i=1}^{n} b_i \right) x + \sum_{i=1}^{n} x^{b_i} + 2 \]
**Proof.** Let \( T(b_1, b_2, \ldots, b_n) \) be the caterpillar tree obtained from the path \( P_{n+2} \) on \( n + 2 \) vertices labelled consecutively as \( u_1, u_2, \ldots, u_n, u_{n+1}, u_{n+2} \), and let \( b_i = |\nu_{1,1}, \nu_{1,2}, \ldots, \nu_{1,b_i}| : 1 \leq i \leq n \) be the pendant vertex set with cardinality \( b_i \) attaching to the vertex \( u_{i+1} \), for \( 1 \leq i \leq n \). Assume, without lose of generality, that \( b_2 \leq b_3 \leq \ldots \leq b_n \). Then \( \Delta(T(b_1, b_2, \ldots, b_n)) = 1 \), \( \Delta(T(b_1, b_2, \ldots, b_n)) = b_1 + 2 \) and \[ \text{Deg}(T(b_1, b_2, \ldots, b_n), 1) = \sum_{i=1}^{n} b_i + 2, \]
\[ \text{Deg}(T(b_1, b_2, \ldots, b_n), b_i + 2) = |\nu_{i+1}|, \text{ for } 1 \leq i \leq n \]
\[ \text{Deg}(T(b_1, b_2, \ldots, b_n), j) = \phi, \text{ for every } j \notin \{ 1, b_i + 2 : 1 \leq i \leq n \} \]

Thus \[ \text{deg}(T(b_1, b_2, \ldots, b_n), 1) = \left( \sum_{i=1}^{n} b_i \right) + 2, \]
\[ \text{deg}(T(b_1, b_2, \ldots, b_n), b_i + 2) = 1, \text{ for every } 1 \leq i \leq n \]
\[ \text{deg}(T(b_1, b_2, \ldots, b_n), j) = 0, \text{ for every } j \notin \{ 1, b_i + 2 : 1 \leq i \leq n \} \]

Therefore, \[ \text{Deg}(T(b_1, b_2, \ldots, b_n), x) = \left( 2 + \sum_{i=1}^{n} b_i \right) x + \sum_{i=1}^{n} x^{b_i} + 2. \]

**Corollary 1.23.** For a caterpillar tree \( T(b, b, \ldots, b) \), the degree polynomial is of the form
\[ \text{Deg}(T(b_1, b_2, \ldots, b_n), x) = (2 + nb)x + nx^{b+2} \]
By token in Definition 1.21, \( n = b \) and \( b_1 = a - 2 \), for \( i = 1, 2, \ldots, n \), we get the caterpillar tree \( T(a, b) \) as defined in [2], and hence we have the following result.

**Theorem 1.24.** For a caterpillar tree \( T(a, b) \), \( a \geq 3 \), the degree polynomial is of the form
\[ \text{Deg}(T(a, b, x) = |(a - 2)| + 2)x + bx^{a'} \]
**Proof.** Let \( T(a, b) \), for \( a \geq 3 \) and \( b \geq 1 \) be the caterpillar tree obtained from the path \( P_{a+2} \) on \( a+2 \) vertices labelled consecutively as \( u_1, u_2, \ldots, u_{a+1}, u_{a+2} \), and let \( a_1 = \{ u_{1,1}, u_{1,2}, \ldots, u_{1,a_1} : 1 \leq i \leq b \} \) be the pendant vertex set with cardinality \( a - 2 \) attaching to the vertex \( u_{i+1} \), for \( 1 \leq i \leq b \).

Then \( \Delta(T(a, b)) = 1 \), \( \Delta(T(a, b)) = a \) and \[ \text{Deg}(T(a, b), 1) = \left| u_{i+1} \right|, \]
\[ \text{Deg}(T(a, b), a) = \{ u_{2}, u_{3}, \ldots, u_{a+1} \}, \]
\[ \text{Deg}(T(a, b), j) = \phi, \text{ for every } 1 < j < a. \]
Thus \[ \text{deg}(T(a, b), 1) = (a - 2) + 2, \text{ deg}(T(a, b), a) = b, \]
and \[ \text{deg}(T(a, b), j) = 0, \text{ for every } 1 < j < a. \]
Therefore, \( \deg(T(a,b),x) = [b(a-2) + 2]x + bx^a \)

**Definition 1.25.** [2] A thorn rod tree, \( P_{p,t} \), is a graph which includes a linear chain (termed rod) of \( p \) vertices and degree \( t \) terminal vertices at each of the two rod ends.

![Fig. 8. A thorn rods \( P_{p,t} \).](image)

**Theorem 1.26.** For a thorn rod tree \( P_{p,t} \), the degree polynomial is of the form

\[
\deg(P_{p,t},x) = 2(t-1)x + (p-2)x^2 + 2x^t
\]

**Proof.** Let \( P_{p,t} \), for \( p, t \geq 2 \) be the thorn rod tree obtained from the path \( P_\ell \) with \( p \) vertices labelled consecutively as \( u_1, u_2, ..., u_p \) by attaching \( t-1 \) vertices, namely \( \{v_1, v_2, ..., v_{t-1}\} \) to the extreme vertex \( u_t \) of the path \( P_\ell \) and also \( t-1 \) vertices, \( v_1, w_1, w_2, ..., w_i, \), to the other extreme vertex \( u_1 \) of the path \( P_\ell \). Then \( \Delta(P_{p,t}) = I \), \( \Delta(P_\ell) = t \), and \( \deg(P_{p,t},1) = \{v_1, v_2, ..., v_{t-1}, w_1, w_2, ..., w_i, \} \), \( \deg(P_{p,t},2) = \{u_1, u_2, ..., u_p\} \), \( \deg(P_{p,t},t) = \{v_1, u_1, u_t\} \) and \( \deg(P_{p,t},i) = \phi \), for every \( 3 \leq i \leq t-1 \).

Thus

\[
\deg(P_{p,t},1) = 2(t-1), \deg(P_{p,t},2) = p-2, \\
\deg(P_{p,t},t) = 2 \text{ and } \deg(P_{p,t},i) = 0, \\
\text{for every } 3 \leq i \leq t-1.
\]

Therefore, \( \deg(P_{p,t},x) = 2(t-1)x + (p-2)x^2 + 2x^t \)

**Definition 1.27.** [2] A thorn star \( S_{k,t} \), for \( k, t \geq 2 \), with \( 1 + kt \) vertices, is a graph obtained from a \( k \)-arm star by attaching \( t-1 \) terminal vertices to each of the star arms.

![Fig. 9. A thorn stars \( S_{k,t} \).](image)

**Theorem 1.28.** For thorn stars \( S_{k,t} \), the degree polynomial is of the form

\[
\deg(S_{k,t},x) = (k-1)x + kx^t + x^k
\]

**Proof.** Let \( S_{k,t} \), for \( k, t \geq 2 \), be the thorn star obtained from a \( k \)-arm star with \( v_0 \) center vertex and \( v_1, v_2, ..., v_k \) as arms vertices by attaching the pendant vertices \( u_1, u_2, ..., u_{tk} \) to the arm vertex \( v_1 \), for every \( 1 \leq i \leq k \). Then

\[
\deg(S_{k,t},1) = \bigcup_{1 \leq j \leq t-1} \{v_j\}, \\
\deg(S_{k,t},k) = \{v_0\}, \\
\deg(S_{k,t},t) = \{v_t : 1 \leq i \leq k\}, \\
\deg(S_{k,t},i) = \phi, \text{ for every } 1 \notin \{1, k, t\},
\]

Thus \( \deg(S_{k,t},1) = \sum_{i=1}^k |u_{ij} : 1 \leq j \leq t-1| \)

\[
= \sum_{i=1}^k (t-1) = k(t-1),
\]

\[
deg(S_{k,t},k) = 1, \deg(S_{k,t},t) = k \text{ and } \\
deg(S_{k,t},i) = 0, \text{ for every } i \notin \{1, k, t\}
\]

Therefore, \( \deg(S_{k,t},x) = k(t-1)x + kx^t + x^k \)

The Kragujevac trees, was proposed in [9]. In order to define the Kragujevac trees, we first explain the structure of its branches.

**Definition 1.29.** [6] Let \( P_3 \) be the 3-vertex tree, rooted at one of its terminal vertices, see Fig.8. For \( k = 2, 3, ..., \), construct the rooted tree \( B_k \) by identifying the roots of \( k \) copies of \( P_3 \). The vertex obtained by identifying the roots of \( P_3 \)-trees is the root of \( B_k \).

![Fig. 10. The rooted trees \( B_2, B_3 \) and \( B_k \).](image)

**Definition 1.30.** [6] Let \( d \geq 2 \) be an integer. Let \( \beta_1, \beta_2, ..., \beta_d \) be rooted trees specified in Definition 1.29, i.e., \( \beta_1, \beta_2, ..., \beta_d \in \{B_2, B_3, ..., \} \). A Kragujevac tree \( T \) is a tree possessing a vertex of degree \( d \), adjacent to the roots of \( \beta_1, \beta_2, ..., \beta_d \). This vertex is said to be the central vertex of \( T \), whereas \( d \) is the degree of \( T \). The subgraphs \( \beta_1, \beta_2, ..., \beta_d \) are the branches of \( T \). Recall that some (or all) branches of \( T \) may be mutually isomorphic.

A typical Kragujevac tree is depicted in Fig.9.

![Fig. 11. A Kragujevac tree of degree \( d = 5 \), in which \( \beta_1, \beta_2, \beta_3 = B_2; \beta_4 = B_3 \) and \( \beta_5 = B_7 \).](image)
\textbf{Theorem 1.31.} Let $b_i$ be the number of $\beta_i$ branches such that $\beta_i \equiv B_i$, $i = 1, 2, \ldots, d$ and $t = \max \{k : \beta_i \equiv B_i\}$. Then the degree polynomial of a Kragujevac tree $T$ of order $n$ and degree $d$, is

$$\text{Deg}(T, x) = \frac{n-d-1}{2} (x + x^2) + x^d + \sum_{i=2}^{t} b_i x^{i+1}. $$

\textbf{Proof.} Let $T$ be a Kragujevac tree on $n$ vertices and degree $d \geq 2$ and branches $\beta_1, \beta_2, \ldots, \beta_d$ such that $\beta_i \equiv B_i$, for $1 \leq i \leq d$ and let $b_i$ be the number of branches $\beta_i$ that $\beta_i \equiv B_i$, and $t = \max \{k : \beta_i \equiv B_i\}$, where $B_i \in \{B_2, B_3, \ldots\}$ . Let we pertaining the vertex set of $T$ into $\{t_{V_0}\} \cup V_0(T) \cup V_1(T) \cup V_2(T)$, where $t_{V_0}$ be the central vertex of $T$ of degree $d$,

$$V_0(T) = \bigcup_{i=1}^{t} \{v_{t_{V_0}} : v_{t_{V_0}} \text{ is the root vertex of branch } \beta_i\},$$

$$V_1(T) = \bigcup_{i=1}^{t} \{v_{t_{V_1}} : v_{t_{V_1}} \in V(\beta_i), \deg(v_{t_{V_1}}) = 1\},$$

$$V_2(T) = \bigcup_{i=1}^{t} \{v_{t_{V_2}} : v_{t_{V_2}} \in V(\beta_i), \deg(v_{t_{V_2}}) = 2\},$$

and since $V_0(T) = \bigcup_{i=1}^{t} \{v_{t_{V_0}} : v_{t_{V_0}} \text{ is the root vertex of branch } \beta_i\}$, $V_1(T) = \bigcup_{i=1}^{t} \{v_{t_{V_1}} : v_{t_{V_1}} \in V(\beta_i), \deg(v_{t_{V_1}}) = 1\}$, $V_2(T) = \bigcup_{i=1}^{t} \{v_{t_{V_2}} : v_{t_{V_2}} \in V(\beta_i), \deg(v_{t_{V_2}}) = 2\}$, such $\beta_i \equiv B_i$,

\begin{align*}
\text{Deg}(T, 1) &= V_1(T), \\
\text{Deg}(T, 2) &= V_2(T), \\
\text{Deg}(T, d) &= \{t_{V_0}\}, \\
\text{Deg}(T, 3) &= \bigcup_{i=1}^{t} V_0(T), \\
\text{Deg}(T, 4) &= \bigcup_{i=1}^{t} V_1(T), \\
\text{Deg}(T, t+1) &= \bigcup_{i=1}^{t} V_2(T), \\
\text{Deg}(T, t+2) &= \bigcup_{i=1}^{t} V_3(T), \\
\text{Deg}(T, t+3) &= \bigcup_{i=1}^{t} V_4(T), \\
\text{Deg}(T, t) &= \bigcup_{i=1}^{t} V_5(T), \\
\text{Deg}(T, t+4) &= \bigcup_{i=1}^{t} V_6(T),
\end{align*}

such $\beta_i \equiv B_i$.

It is clear from the definition of the Kragujevac tree $T$, that $\text{Deg}(T, 1) = \text{Deg}(T, 2)$. Thus,

$$V(T) = V_0(T) \cup V_1(T) \cup \text{Deg}(T, 1) \cup \text{Deg}(T, 2)$$

and since $|V_0(T)| = d$ and $\text{Deg}(T, 1) = \text{Deg}(T, 2)$ then

$$n = 1 + d + 2 \text{deg}(T, 1), \text{ i.e., } \text{deg}(T, 1) = \text{deg}(T, 2) = \frac{n-d-1}{2}.$$ 

Also we have $\text{deg}(T, d) = 1$, $\text{deg}(T, 3) = b_2$, $\text{deg}(T, 4) = b_3$, ..., $\text{deg}(T, t+1) = b_t$.

Therefore,

$$\text{Deg}(T, x) = \frac{n-d-1}{2} x + \frac{n-d-1}{2} x^2 + b_2 x^3 + b_3 x^4 + \ldots + b_t x^{t+1} + \sum_{i=2}^{t} b_i x^{i+1} + x^d.$$ 

\section*{II. Conclusion}

In this paper, we extended the study of the degree polynomial $\text{Deg}(G, x)$ of graphs. We computed The degree polynomial of some well-known families of tree. It turned out that the constant term of the degree polynomial of any tree is zero and hence the zero is a degree root of every tree. There are a lot of problems in this concepts for future study, we mention some of them as follows:

1. Calculate the degree roots of every tree mentioned to it in the present paper.
2. Classification all trees with only integers degree roots.

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\section*{REFERENCES}