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A Study of W_2-Curvature Tensors in K-Contact Riemannian Manifold

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Abstract **– In this paper we study the W_2 –curvature tensor. We consider the different characterisations of the W_2 –curvature tensor in W_2 –symmetric and W_2 –flat k-contact Riemannian manifold.**

Keywords **– K-contact Riemannian manifold; W_2-symmetric and W_2-flat.**

I. Preliminaries

Let M^n be an $n(=2m+1)$ dimensional almost contact metric manifold with the structure $(\emptyset,\xi,\eta,g)$, where $\emptyset$ is a (1,1) tensor field, ξ is a vector field, η a 1-form and g a Riemannian metric. Let $X,Y,Z,V,...$ be the vector fields on M^n. Then the following is valid on M:

(1.1) $\emptyset^2 x = -x + \eta(x)\xi, \quad \eta(\xi) = 1$
$\emptyset(\xi) = 0 \quad \xi = 0 \quad \eta \otimes \quad \emptyset = 0.$

(1.2) $g(\emptyset X, \emptyset Y) = -g(X,Y) = \eta(X)\eta(Y)$

(1.3) $(a)\ g(X,\emptyset Y) = -g(\emptyset X, Y)$
$(b)\ g(X,\xi) = \eta(X)$ (see[1]).

An almost contact metric manifold is said to be a contact metric manifold if

(1.4) $d\eta(X,Y) = g(X,\emptyset Y)$.

An almost contact manifold is said to be k-contact manifold if

(1.5) $\nabla\xi = -\emptyset$.

If $\nabla_X \xi = -\emptyset X - \emptyset \bar{h} X$ (see [1] and [2]), where $\bar{h} = \frac{1}{2} L_\xi \emptyset$ is the lie derivative, but in k-contact $\bar{h}\xi = 0 \Rightarrow \nabla_X \xi = -\emptyset X$.

An almost contact metric manifold is a Sasakian manifold if;

(1.6) $(\nabla_X \emptyset)Y = g(X,Y)\xi - \eta(Y)\eta(X)$ (see [3])

A contact metric manifold is Sasakian if and only if;

(1.7) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$.

In a $n(=2m+1)$-dimensional contact metric manifold M, if $\{e_{1,\dots e_{2n,\xi}}\}$ is a local orthonormal basis of vector fields on M, then $\emptyset e_{1,\dots,\emptyset e_{2n},\xi}$ is also a local orthonormal basis. It is easy to verify that;

(1.8) $\sum_{i=1}^{2n} g(e_i,e_i) = \sum_{i=1}^{2n} g(\emptyset e_i, \emptyset e_i) = 2n$

(1.9) $\sum_{i=1}^{2n} g(e_i,Z)S(Y,e_i) = \sum_{i=1}^{2n} g(\emptyset e_i, Z)S(Y,\emptyset e_i) = S(Y,Z) - S(Y,\xi)n(z)$,

where $S(Y,Z) = Ric(Y,Z)$ i.e. S is the Ricci tensor. If M is a k-contact manifold then we have,

(1.10) $S(X,\xi) = 2n\eta(X)$,

(1.11) $(\nabla_X \emptyset)Y = -R(X,\xi)Y$

(1.12) $(\nabla_X \emptyset)(\emptyset Y) + \emptyset(\nabla_X \emptyset)Y = -g(\emptyset X, Y)\xi - \eta(Y)\emptyset X$

Then from (1.11) and (1.12), we get

(1.13) $\emptyset R(X,\xi)Y + R(X,\xi)\emptyset Y = g(\emptyset X, Y)\xi - \eta(Y)\emptyset X$ and

(1.14) $R(X,\xi)\xi = X - \eta(X)\xi$

(1.15) $Ric(\xi,\xi) = 2n$

From (1.15), we get

(1.16) $\sum_{i=1}^{2n} S(e_i,e_i) = \sum_{i=1}^{2n} S(\emptyset e_i, \emptyset e_i) = r - 2n$

where r is the scalar curvature in a k-contact manifold. We also get,

(1.17) $R(\xi,Y,Z,\xi) = g(\emptyset Y, \emptyset Z)$.

Hence,

(1.18) $\sum_{i=1}^{2n} R(e_i,Y,Z,e_i) = \sum_{i=1}^{2n} R(\emptyset e_i, Y, Z, \emptyset e_i) - g(\emptyset Y, \emptyset Z)$.

II. W_2-Symmetric K-Contact Riemannian Manifold

Mishra and Pokhariyal [5] defined the W_2-Curvature tensors as;

$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U)]$

The k-Contact Riemannian manifold is said to be W_2-Symmetric if,

$\nabla_X W_2(Y,Z)U = 0$, that is $\nabla_X W_2(Y,Z)U = W_2(X,Y,Z)U = 0$.

Theorem 2.1.

A W_2-Symmetric k-Contact Riemannian manifold is a space of constant scalar curvature.

Proof.

$\nabla_U W_2(X,Y)Z = W_2(X,Y,Z) = 0 = R'(X,Y,Z,U) + \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(X,Z)Ric(Y,U)]$

Or

$R(X,Y,Z,U) = \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(X,Z)Ric(Y,U)]$, but

$Ric(X,U) = (n-1)g(X,U)$ from (1.15)

$$\Rightarrow R(X,Y,Z,U) = \frac{1}{n-1}[(n-1)g(Y,Z)g(X,U) - (n-1)g(X,Z)g(Y,U)] = g(Y,Z)g(X,U) - g(X,Z)g(Y,U)$$

Or

$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$, which on contracting and setting $X = Y = \xi$, we have $Ric(\xi,\xi) = n - 1 = 2m = S(\xi,\xi)$ since $n = 2m+1$. Hence the result.

Corollary 1:

A W_2-Symmetric k-Contact Riemannian manifold is not a flat space.

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Corollary 2:

A W_2-Symmetric k-Contact Riemannian manifold is not η-Einstein manifold.

Theorem 2.2.

A W_2-Symmetric k-Contact Riemannian manifold is W_2-flat manifold.

Proof.

From $\nabla_X W_2(Y,Z,U,V) = W_2(X,Y,Z,U,V) = 0$

We have,

(2.2.1) $R(X,Y,W_2(Z,U,V)) - W_2(R(X,Y,Z),U,V) - W_2(Z,R(X,Y,U),V) - W_2(Z,U,R(X,Y,V)) = 0$

Or

(2.2.2) $R'(X,Y,W_2(Z,U,V),\xi) - W_2'(R(X,Y,Z),U,V,\xi) - W_2'(Z,R(X,Y,U),V,\xi) - W_2'(Z,U,R(X,Y,V),\xi) = 0$

Let $X = \xi$ on equation (2.2.2) and expanding each and every term on it we have,

(2.2.3) $R'(\xi,Y,W_2)(Z,U,V),\xi) = g(\bar{Y},\overline{W_2}(Z,U,V) = g(Y,W_2(Z,U,V) - \eta(Y)\eta(W_2(Z,U,V)) = W_2'(Y,Z,U,V) - \eta(Y)\eta(W_2(Z,U,V))$.

(2.2.4) $W_2'(R(X,Y,Z),U,V,\xi) = R'(R(\xi,Y,Z),U,V,\xi) + \frac{1}{n-1}[g(R(\xi,Y,Z),V)Ric(U,\xi) - g(U,V)Ric(R(\xi,Y,Z),\xi)]$

$= R'(R(\xi,Y,Z),U,V,\xi) + \frac{1}{n-1}[(n-1)\eta(U)g(R(\xi,Y,Z) - (n-1)g(U,V)\eta(R(\xi,Y,Z),U)]$

$= R'(R(\xi,Y,Z),U,V,\xi) + \eta(U)g(R(\xi,Y,Z),V) - g(U,V)\eta(R(\xi,Y,Z)$

$= g(U,V)\eta(R(\xi,Y,Z) - \eta(U)g(R(\xi,Y,Z),V) + \eta(U)g(R(\xi,Y,Z),V) - g(U,V)\eta(R(\xi,Y,Z) = 0$

(2.2.5) $W_2'(Z,R(X,Y,U),V,\xi)) = R'(Z,R(\xi,Y,U),V,\xi))) + \frac{1}{n-1}[g(Z,V)Ric(R(\xi,Y,V),\xi)) - g(R(\xi,Y,U),V)Ric(Z,\xi)]$

$$= R'(Z,R(\xi,Y,U),V,\xi) + \frac{1}{n-1}[(n-1)g(Z,V)\eta(R(\xi,Y,U)) - (n-1)\eta(Z)g(R(\xi,Y,U),V)]$$

$$= R'(Z,R(\xi,Y,U),V,\xi) + g(Z,V)\eta(R(\xi,Y,U)) - \eta(Z)g(R(\xi,Y,U),V)$$
$$= g(R(\xi,Y,U),V)\eta(Z) - g(Z,V)\eta(R(\xi,Y,U) + g(Z,V)\eta(R(\xi,Y,Z) - \eta(Z)g(R(\xi,Y,U),V) = 0$$

(2.2.6) $W_2'(Z,R(X,Y,U),V,\xi))$

$$= R'(Z,U,R(\xi,Y,V),\xi))) + \frac{1}{n-1}[g(R(\xi,Y,V),Z)(Ric(U,\xi) - g(U,R(\xi,R(\xi,Y,V)Ric(Z,\xi)]$$

$$= R'(Z,U,R(\xi,Y,V),\xi))) + [\eta(U)g(R(\xi,Y,V)) - \eta(Z)g(R(U,R(\xi,Y,V))]$$
$$= g(U,R(\xi,Y,V))\eta(Z) - g(Z,R)((\xi,Y,V)\eta(V) + \eta(U)g(R(R(\xi,Y,V) - \eta(Z)g(U,R((\xi,Y,V)) = 0$$

From (2.2.3), (2.2.4), (2.2.5) and (2.2.6), we have $W_2'(Y,Z,U,V) - \eta(Y)\eta(W_2(Z,U,V) = 0 \Longrightarrow W_2'(Y,Z,U,V) = 0$.

Since $\eta(W_2'(Y,Z,U,V)) = gW_2(Z,V,Z) = 0$. Hence the theorem.

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