International Journal of Innovation in Science and Mathematics
Volume 5, Issue 6, ISSN (Online): 2347–9051

Proof of Existence and Uniqueness of Solutions in Epidemiology: Case of SI Model

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Date of publication (dd/mm/yyyy): 13/11/2017

Abstract **– In this paper we propose a proof of existence and uniqueness of solutions for SI model in dimension 2. We present firstly the variational formulation by introducing a Sobolev spaces and then we give the statement of Lions theorem.**

Keywords **– SI Model, Spread of Epidemics, Sobolev Spaces, Partial Differential Equation.**

I. INTRODUCTION

The Kermack-McKendrick model is a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population. After Kermack-McKendrick model, different epidemic models have been proposed and studied in the literature (see Capasso and Serio [3], Hethcote and Tudor [4], Liu [5][6], Hethcote and van den Driessche [8], Derrick and van den Driessche [7], Beretta and Takeuchi [12][11], Ma et al. [9][10], Song et al. [13]. In 1906 appears the first dynamic model of W.H. Hamer (SI model).

The SI Model is the simplest one among the epidemic models. That is why it is also called the Simple Model. We divide the population just in the susceptible compartment S(t) and the infectious compartment I(t). We do assume the disease to be highly infectious but not serious, which means that the infectives remain in contact with susceptibles for all time t > 0. We also assume that the infectives continue to spread the disease till the end of the epidemic, the population size to be constant (S(t) + I(t) = N) and homogeneous mixing of population. Infection rate is proportional to the number of infectives. The remaining parts of this paper are organized as follows: section 2 presents the variational analysis for the deterministic SI model by introducing a Sobolev spaces and the proof of existence and uniqueness of solutions for SI model in dimension 2. The last section provides concluding remarks.

II. DISSEMINATION OF EPIDEMIC FOR SI MODEL IN DIMENSION 2

In this part we give a variational analysis for the deterministic SI model and their link with the partial differential equations: the problem is considered in the framework of weighted Sobolev spaces. We consider the system:

$$\begin{cases} \dot{S}(t) &= l_1\dfrac{\partial^2 S}{\partial x^2}+l_1\dfrac{\partial^2 S}{\partial y^2}+r_cS\left(1-\dfrac{S}{k}\right)-\dfrac{\alpha SI}{1+a_1I+a_2S+a_3SI} \\ \dot{I}(t) &= l_2\dfrac{\partial^2 I}{\partial x^2}+l_2\dfrac{\partial^2 I}{\partial y^2}+\dfrac{\alpha SI}{1+a_1I+a_2S+a_3SI}-(\gamma+\beta)I \end{cases} \quad (1)$$

The model has a susceptible group designated by S, and an infected group I, rc is the intrinsic growth rate of susceptible, k is the carrying capacity of the susceptible in the absence of infective, l1, l2 are the diffusion coefficients, α is the maximum values of per capita reduction rate of S du to I, a1, a2 and a3 are half saturation constants, β is the death rate of infected populations and γ is the natural recover rate from infection.

This problem can be rewritten as follows:

$$\begin{cases} \text{Find } u=(S,I) \text{ such that :} \\ \dfrac{\partial S}{\partial t}(t,x,y)-\mathcal{L}_1S(t,x,y) = 0 \quad \forall t\in[0,T] \text{ and } (x,y)\in\mathbb{R}^2 \\ \dfrac{\partial I}{\partial t}(t,x,y)-\mathcal{L}_2I(t,x,y) = 0 \\ u(0,x,y) = h(x) \qquad (x,y)\in\mathbb{R}^2 \end{cases} \quad (2)$$

With:

$$\begin{aligned} \mathcal{L}_1 &= l_1\frac{\partial^2 S}{\partial x^2}+l_1\frac{\partial^2 S}{\partial y^2}+r_cS\left(1-\frac{S}{k}\right)-\frac{\alpha SI}{1+a_1I+a_2S+a_3SI} \\ \mathcal{L}_2 &= l_2\frac{\partial^2 I}{\partial x^2}+l_2\frac{\partial^2 I}{\partial y^2}+\frac{\alpha SI}{1+a_1I+a_2S+a_3SI}-(\gamma+\beta)I \end{aligned} \quad (3)$$

We will use the variational formulation in order to establish the existence and uniqueness.

2.1. Variational Formulation

To give a variational formulation, we introduce some weighted Sobolev spaces, we denote by $U\subset\mathbb{R}^2$ a bounded domain.

$L^2(U)$ is a space of measurable functions u and 2^{th} integrable.

This space is equipped with the norm $\| u \|_{L^2(U)} := |u| = \left(\int_U |u|^2\right)^{1/2}$.

And the $L^2(U)$ inner product denoted by $(.,.)_{L^2}$ and defined as follows:

$$(u,v)_{L^2} = \int_U uv$$

$W^{1,2}$ is the space of functions u in $L^2(U)$ such that the weak partial derivative $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ belong to $L^2(U)$, equipped with the norm:

$$\|u\|_{W^{1,2}(U)} := \|u\| = \left(\int_U |u|^2 + \int_U \sum_{i=1}^{2}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{1/2}.$$

Let

$$H = \{u \big| (u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, Au, Bu, Du) \in (L^2(\Omega))^6\}$$

With $A = \sqrt{\frac{I}{1+a_1I+a_2S+a_3SI}}$, $B = \sqrt{\frac{S}{1+a_1I+a_2S+a_3SI}}$ and $D = \sqrt{S}$

this space equipped the norm :

$$\|u\|_H = \int \left(u^2 + (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + A^2u^2 + B^2u^2 + D^2u^2\right)^{\frac{1}{2}}$$

And

$$\|u\|_H = \left(\|u\|_{L^2}^2 + \|\frac{\partial u}{\partial x}\|_{L^2}^2 + \|\frac{\partial u}{\partial y}\|_{L^2}^2 + A^2\|u\|_{L^2}^2 + B^2\|u\|_{L^2}^2 + D^2\|u\|_{L^2}^2\right)^{\frac{1}{2}}$$

We put:

$$\|u\|_H^2 = (\|u\|_1^2 + \|u\|_2^2 + \|u\|_3^2 + \|u\|_4^2)$$

$$\|u\|_H^2 = (\|u\|_1^2 + \|u\|_2^2 + \|u\|_3^2 + \|u\|_4^2)$$

with:

$\|u\|_1$ the norm of Sobolev space, $\|u\|_2 = A\|u\|_{L^2(\Omega)}$, $\|u\|_3 = B\|u\|_{L^2(\Omega)}$ and $\|u\|_4 = D\|u\|_{L^2(\Omega)}$

Let $v_1, v_2 \in \mathcal{D}(U)$ ($\mathcal{D}$ is the space of smooth, compactly supported test functions).

We obtain the variational formulation by multiplying the PIDE (2) by (v_1, v_2):

$$\begin{cases} (\frac{\partial S}{\partial t}, v_1) + l_1\int\frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x} + l_1\int\frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y} - \int[r_cS(1-\frac{S}{k}) - \alpha\frac{SI}{1+a_1I+a_2S+a_3SI}]v_1 = 0 \\ (\frac{\partial I}{\partial t}, v_2) + l_2\int\frac{\partial I}{\partial x}\frac{\partial v_2}{\partial x} + l_2\int\frac{\partial I}{\partial y}\frac{\partial v_2}{\partial y} - \int[\alpha\frac{SI}{1+a_1I+a_2S+a_3SI} + (\beta+\gamma)I]v_2 = 0 \end{cases}$$

We consider the Dirichlet boundary conditions. By using the Green-formula,
We obtain

$$\begin{cases} (\frac{\partial S}{\partial t}, v_1)_\alpha + a_1(S, v_1) = 0 & \forall\, t \in [0,T] \text{ and } (x,y) \in U \\ (\frac{\partial I}{\partial t}, v_2)_\alpha + a_2(I, v_2) = 0 & \forall\, t \in [0,T] \text{ and } (x,y) \in U \\ (u(0,.,.), v)_\alpha = (h, v)_\alpha & \forall\,(x,y) \in U \end{cases} \quad (4)$$

Such that

$$(\mathcal{L}_1S, v_1) = a_1(S, v_1).$$
$$(\mathcal{L}_2I, v_2) = a_2(I, v_1)$$

Where:

$$a_1(S, v_1) := l_1\int\frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x} + l_1\int\frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y} - \int[r_cS(1-\frac{S}{k}) - \alpha\frac{SI}{1+a_1I+a_2S+a_3SI}]v_1$$

$$a_2(I, v_2) := l_2\int\frac{\partial I}{\partial x}\frac{\partial v_2}{\partial x} + l_2\int\frac{\partial I}{\partial y}\frac{\partial v_2}{\partial y} - \int[\alpha\frac{SI}{1+a_1I+a_2S+a_3SI} + (\gamma+\beta)I]v_2$$

2.2. Existence and Uniqueness of the Solutions of Variational Problem

Theorem 2.1. Let be $h \in H$.

Then $\exists\, C_1, C_2, C_3$, and C_4 four positive reals such that:

$$|a_1(S, v_1)| \le C_1\|S\|\|v_1\| \qquad \forall S, v_1 \in W^{1,2}$$
$$|a_2(I, v_2)| \le C_2\|I\|\|v_2\| \qquad \forall I, v_2 \in W^{1,2}$$

And there are a positive real ϑ such that

$$a_1(S,S) + \vartheta_1|S|^2 \ge C_3\|S\|^2 \qquad \forall S \in W^{1,2}$$
$$a_2(I,I) + \vartheta_2|I|^2 \ge C_4\|I\|^2 \qquad \forall I \in W^{1,2}$$

Therefore, for $h \in H \cap L^\infty$, the variational problem admits a unique solution in $H \cap L^\infty$. This solution has the probabilistic representation.

2.3. Proof

We will show the continuity and coercivity of the bilinear from a (.,.).

Continuity

We show in a first step the continuity of the bilinear form a1, we have:

$$\begin{aligned} |a_1(S, v_1)| &= \left|l_1\int\frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x} + l_1\int\frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y} - \int[r_cS(1-\frac{S}{k}) - \alpha\frac{SI}{1+a_1I+a_2S+a_3SI}]v_1\right| \\ &\le l_1\left|\int\frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x}\right| + l_1\left|\int\frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y}\right| + \left|\int[r_cS(1-\frac{S}{k}) - \alpha\frac{SI}{1+a_1I+a_2S+a_3SI}]v_1\right| \\ &\le l_1\left\|\frac{\partial S}{\partial x}\right\|_{L^2}\left\|\frac{\partial v_1}{\partial x}\right\|_{L^2} + l_1\left\|\frac{\partial S}{\partial y}\right\|_{L^2}\left\|\frac{\partial v_1}{\partial y}\right\|_{L^2} + r_c\left|\int Sv_1\right| + \frac{r_c}{k}\left|\int S^2v_1\right| + \alpha\int\left|A^2Sv_1\right| \\ &\le 2l_1\|S\|_H\|v_1\|_H + r_c\|S\|_{L^2}\|v_1\|_{L^2} + \frac{r_c}{k}\|S^2\|_{L^2}\|v_1\|_{L^2} + \alpha|S|_2|v_1|_2 \\ &\le 2l_1\|S\|_H\|v_1\|_H + r_c\|S\|_H\|v_1\|_H + \frac{r_c}{k}\|S\|_H\|v_1\|_H + \alpha\|S\|_H\|v_1\|_H \\ &\le (2l_1 + r_c + \frac{r_c}{k})\|S\|_H\|v_1\|_H \end{aligned}$$

Let $C_1 = 2l_1 + r_c + \frac{r_c}{k}$

Now show the continuity of the bilinear form a_2, we have:

$$\begin{aligned} |a_2(I, v_2)| &= \left|l_2\int\frac{\partial I}{\partial x}\frac{\partial v_2}{\partial x} + l_2\int\frac{\partial I}{\partial y}\frac{\partial v_2}{\partial y} - \int[\alpha\frac{SI}{1+a_1I+a_2S+a_3SI} + (\gamma+\beta)I]v_2\right| \\ &\le 2l_2\|I\|_H\|v_2\|_H + \alpha\int\frac{SI}{1+a_1I+a_2S+a_3SI}v_2 + (\gamma+\beta)\int Iv_2 \\ &\le 2l_2\|I\|_H\|v_2\|_H + \alpha|I|_3|v_2|_3 + (\gamma+\beta)\|I\|_{L^2}\|v_2\|_{L^2} \\ &\le 2l_2\|I\|_H\|v_2\|_H + (\alpha+\gamma+\beta)\|I\|_H\|v_2\|_H \\ &\le (2l_2+\alpha+\gamma+\beta)\|I\|_H\|v_2\|_H \end{aligned}$$

Let $C_2 = 2l_2 + \alpha + \gamma + \beta$

Coercivity

- Coercivity of a_1

We note:

$$\begin{aligned} \|u\|_1 &= \|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 \\ &= \|u\|_{1,1}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial u}{\partial y}\right\|_{1,3}^2 \end{aligned}$$

$$\|S\|_{1,1} \sim \|S\|_2, \|S\|_{1,1} \sim \|S\|_3, \|S\|_{1,1} \sim \|S\|_4$$

Then:

$$a_1(S,S) + (\frac{1}{l_1} + r_c)\|S\|_{1,1}^2 \ge \frac{1}{l_1}\|S\|_H^2 - (\frac{3}{l_1} + \frac{r_c}{k} + \alpha)\|S\|_{1,1}^2$$

So $\exists \xi_1 > 0$, $\xi_2 > 0$ such that:

$$a_1(S,S) + (\frac{1}{l_1} + r_c)\|S\|_{1,1}^2 \;\geq\; \frac{1}{2l_1}\|S\|_H^2 + \frac{1}{2l_1}\|S\|_H^2 - \xi_1\|S\|_H\|S\|_{1,1} - \xi_2\|S\|_{1,1}^2$$

Let's the quadratic form $:(x,y) \rightarrow \frac{1}{4}x^2 - \xi_1 xy - \xi_2 y^2$ associated to the bilinear form a_1.

Using the classical relation, $AB < \frac{\theta}{2}A^2 + \frac{1}{2\theta}B^2$

We have

$$-\xi_1\|S\|_H\|S\|_{1,1} > -\frac{\xi_1}{2}\theta\|S\|_H^2 - \frac{\xi_1}{2\theta}\|S\|_{1,1}^2$$

With implies that

$$\tfrac{1}{2}\|S\|_H^2 - \xi_1\|S\|_H\|S\|_{1,1} - \xi_2\|S\|_{1,1}^2 \geq \frac{\frac{1}{l_1} - \xi_1\theta}{2}\|S\|_H^2 - \xi_2 - \frac{\xi_1}{2\theta}\|S\|_{1,1}^2$$

Let $S = v_1 \in H$

$$\begin{aligned} a_1(S,S) &= l_1\int\left(\frac{\partial S}{\partial x}\right)^2 + l_1\left(\int\frac{\partial S}{\partial y}\right)^2 - \int r_c S^2 - \int\frac{r_c}{k}S^3 - \alpha\int\frac{S^2 I}{1+a_1 I + a_2 S + a_3 SI} \\ &= l_1\left\|\frac{\partial S}{\partial x}\right\|_{1,2}^2 + l_1\left\|\frac{\partial S}{\partial y}\right\|_{1,3}^2 - r_c\|S\|_{1,1}^2 - \frac{r_c}{k}\|S\|_4^2 - \alpha\|S\|_2^2 \end{aligned}$$

So

$$a_1(S,S) + r_c\|S\|_{1,1}^2 \;=\; -\frac{r_c}{k}\|S\|_4^2 - \alpha\|S\|_2^2 + l_1\left\|\frac{\partial S}{\partial x}\right\|_{1,2}^2 + l_1\left\|\frac{\partial S}{\partial y}\right\|_{1,3}^2$$

We know that:

$$\left\|\frac{\partial S}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial S}{\partial y}\right\|_{1,3}^2 \;=\; \|S\|_H^2 - \|S\|_{1,1}^2 - \|S\|_2^2 - \|S\|_3^2 - \|S\|_4^2$$

So

$$a_1(S,S) + (\frac{1}{l_1} + r_c)\|S\|_{1,1}^2 \;=\; \frac{1}{l_1}\|S\|_H^2 - (\frac{1}{l_1} + \frac{r_c}{k})\|S\|_4^2 - (\frac{1}{l_1} + \alpha)\|S\|_2^2 - \frac{1}{l_1}\|S\|_3^2$$

Using the equivalence between the semi norms H:

Taking $\theta = \frac{1}{\xi_1}$ and $\xi_3 > 0$ such that $\xi_3 - \xi_2 - \frac{\xi_1}{2\theta} > 0$

Then

$$\frac{1}{l_1}\|S\|_H^2 - \xi_1\|S\|_H\|S\|_{1,1} - \xi_2\|S\|_{1,1}^2 \geq (\xi_3 - \xi_2 + \frac{-\xi_1}{2\theta})\|S\|_{1,1}^2 \geq 0$$

So we have:

$$a_1(S,S) + \vartheta_1|S|^2 \geq C_3\|S\|^2$$

With

$$\begin{aligned} \vartheta_1 &= \frac{l}{l_1} + r_c \\ C_3 &= \frac{1}{2l_1} \end{aligned}$$

- Coercivity of a_2

For $I = v_2 \in H$ we have

$$\begin{aligned} a_2(I,I) &= l_2\int\left(\frac{\partial I}{\partial x}\right)^2 + l_2\int\left(\frac{\partial I}{\partial y}\right)^2 - \int\alpha\frac{SI^2}{1+a_1 I + a_2 S + a_3 SI} - \int(\gamma+\beta)I^2 \\ &= l_2\left\|\frac{\partial I}{\partial x}\right\|_{1,2}^2 + l_2\left\|\frac{\partial I}{\partial y}\right\|_{1,3}^2 - \alpha\|I\|_3^2 - (\gamma+\beta)\|I\|_4^2 \end{aligned}$$

We know that:

$$\left\|\frac{\partial I}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial I}{\partial y}\right\|_{1,3}^2 \;=\; \|I\|_H^2 - \|I\|_{1,1}^2 - \|I\|_2^2 - \|I\|_3^2 - \|I\|_4^2$$

So

$$a_2(I,I) + \frac{1}{l_2}\|I\|_{1,1}^2 \;=\; \frac{1}{l_2}\|I\|_H^2 - \frac{1}{l_2}\|I\|_2^2 - (\frac{1}{l_2} + \alpha)\|I\|_3^2 - (\frac{1}{l_2} + \gamma + \beta)\|I\|_4^2$$

Using the equivalence between the semi norms H:

$$\|I\|_{1,1} \sim \|I\|_2, \|I\|_{1,1} \sim \|I\|_3, \|I\|_{1,1} \sim \|I\|_4$$

Then:

$$a_2(I,I) + \frac{1}{l_2}\|I\|_{1,1}^2 \;\geq\; \frac{1}{l_2}\|I\|_H^2 - (\frac{3}{l_2} + \alpha + \gamma + \beta)\|I\|_{1,1}^2$$

So $\exists \xi_3 > 0, \xi_4 > 0$ such that:

$$a_2(I,I) + \frac{1}{l_2}\|I\|_{1,1}^2 \;\geq\; \frac{1}{2l_2}\|I\|_H^2 + \frac{1}{2l_2}\|I\|_H^2 - \xi_3\|I\|_H\|I\|_{1,1} - \xi_4\|I\|_{1,1}^2$$

Let's the quadratic form $:(x,y) \rightarrow \frac{1}{4}x^2 - \xi_3 xy - \xi_4 y^2$ associated to the bilinear form a_2.

By the same reasoning like a_1 we obtained:

$$a_2(I,I) + \vartheta_2|I|^2 \geq C_4\|I\|^2$$

With

$$\begin{aligned} \vartheta_2 &= \frac{1}{l_2} \\ C_4 &= \frac{1}{2l_2} \end{aligned}$$

III. Conclusion

In this work we have studied an epidemiological model in dimension 2 and we proposed a proof of existence and uniqueness of solutions for SI model by introducing a Sobolev spaces.

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