International Journal of Innovation in Science and Mathematics
Volume 6, Issue 1, ISSN (Online): 2347–9051

General & Traveling Wave Solutions of Non-Linear Schrodinger (NLS) PDEs

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Date of publication (dd/mm/yyyy): 05/02/2018

***Abstract* – Nonlinear (non-linear) Schrodinger (NLS) PDEs are very important in applications including wave guidance, optics, quantum mechanics, and more found in physics, electrical engineering, and industrial applied mathematics. This article covers some practical problem-solution examples of NLS PDEs through definitions, then corresponding examples. The function u is assumed complex for real variables x and t as well as y as appropriate.**

***Keywords* – Schrodinger, PDEs, Nonlinear, NLS, Differential Equations, Evolution, Wave, Quantum.**

I. Introduction

This article is a review of the various forms of NLS equations and the kinds of solutions these equations emit based on definitions from recent literature and then author-driven examples. The function u is a wave function and is assumed complex for real variables x and t as well as y as appropriate. The NLS equations and the examples provided below will have arbitrary real constants and/or arbitrary real functions of real variables.

Definition 1:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and k is a real constant in this one dimensional (1D) nonlinear Schrodinger (NLS) equation.

$$iu_t + u_{xx} + k\left|u\right|^2 u = 0$$

With the following general solutions:

$$u(x,t) = C_1 exp(i[C_2 x + (kC_1^2 - C_2^2)t + C_3])$$

$$u(x,t) = C_1\sqrt{\frac{2}{k}}\frac{exp[i(C_1^2 t + C_2)]}{cosh(C_1 x + C_3)}$$

$$u(x,t) = \frac{C_1}{\sqrt{t}} exp[i\frac{(x+C_2)^2}{4t} + i(kC_1^2 ln(t) + C_3)]$$

Source: Polyanin (2004).

Example 1:

With k = 3 in above NLS equation, we get the following:

$$iu_t + u_{xx} + 3\left|u\right|^2 u = 0$$

Using above definition with k = 3, we get the following general solutions:

$$u(x,t) = C_1 exp(i[C_2 x + (3C_1^2 - C_2^2)t + C_3])$$

$$u(x,t) = C_1\sqrt{\frac{2}{3}}\frac{exp[i(C_1^2 t + C_2)]}{cosh(C_1 x + C_3)}$$

$$u(x,t) = \frac{C_1}{\sqrt{t}} exp[i\frac{(x+C_2)^2}{4t} + i(3C_1^2 ln(t) + C_3)]$$

Definition 2:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and A and B are real constants in this one dimensional (1D) nonlinear Schrodinger (NLS) equation.

$$iu_t + u_{xx} + (A\left|u\right|^2 + B)u = 0$$

With the following general solutions:

$$u(x,t) = C_1 exp(i[C_2 x + (AC_1^2 + B - C_2^2)t + C_3])$$

$$u(x,t) = \frac{C_1}{\sqrt{t}} exp[\frac{i(x+C_2)^2}{4t} + i(AC_1^2 ln(t) + Bt + C_3]$$

Source: Polyanin (2004).

Example 2:

With A = 4, B = 6 in above NLS equation, we get:

$$iu_t + u_{xx} + (4\left|u\right|^2 + 6)u = 0$$

Using above definition and these values, we get the following general solutions:

$$u(x,t) = C_1 exp(i[C_2 x + (4C_1^2 + 6 - C_2^2)t + C_3])$$

$$u(x,t) = \frac{C_1}{\sqrt{t}} exp[\frac{i(x+C_2)^2}{4t} + i(4C_1^2 ln(t) + 6t + C_3]$$

Definition 3:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and A and n are real constants in this one dimensional (1D) nonlinear Schrodinger (NLS) equation.

$$iu_t + u_{xx} + A\left|u\right|^{2n} u = 0$$

With the following solutions:

$$u(x,t) = C_1 exp(i[C_2 x + (A\left|C_1\right|^{2n} - C_2^2)t + C_3])$$

$$u(x,t) = \pm[\frac{(n+1)C_1^2}{Acosh^2(C_1 nx + C_2)}]^{1/2n} exp[i(C_1^2 t + C_3)]$$

$$u(x,t) = \frac{C_1}{\sqrt{t}} exp[i\frac{(x+C_2)^2}{4t} + i\frac{AC_1^{2n}}{1-n}t^{1-n} + C_3]$$

Source: Polyanin (2004), M.J. Ablowitz and H. Segur (1981).

Example 3:

With n = 6 and A having the value below:

$$A = 4C_1^2, n = 6$$

We get the following NLS equation:

$$iu_t + u_{xx} + 4C_1^2\left|u\right|^6 u = 0$$

With the following general solutions:

$$u(x,t) = C_1 exp(i[C_2 x + (4C_1^2\left|C_1\right|^6 - C_2^2)t + C_3])$$

$$u(x,t) = \pm[\frac{1}{cosh^2(3C_1 x + C_2)}]^{1/6} exp[i(C_1^2 t + C_3)]$$

$$u(x,t) = \frac{C_1}{\sqrt{t}} exp[i\frac{(x+C_2)^2}{4t} - i\frac{2C_1^8}{t^2} + C_3]$$

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Definition 4:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and A and n are real constants in this 1D nonlinear Schrodinger (NLS) equation.

$$iu_t + \frac{1}{x}(x^n u_x)_x + A|u|^2 u = 0$$

With the following general solution:

$$u(x,t) = C_1(t+C_2)^{(-n+1)/2} exp[i(\frac{x^2}{4(t+C_2)} - \frac{AC_1^2}{n(t+C_2)^n} + C_3)]$$

Source: Polyanin (2004).

Example 4:

With A = 3 and n = -3 in above NLS PDE in definition, we get the following:

$$iu_t + x^3(\frac{1}{x^3}u_x)_x + 3|u|^2 u = 0$$

We get the following general solution:

$$u(x,t) = C_1(t+C_2)^2 exp[i(\frac{x^2}{4(t+C_2)} + C_1^2(t+C_2)^3 + C_3)]$$

Definition 5:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and A, B and n are real constants in this 1D nonlinear Schrodinger (NLS) equation.

$$iu_t + \frac{1}{x^n}(x^n u_x)_x + (A|u|^2 + B)u = 0$$

With the following solution:

$$u(x,t) = C_1(t+C_2)^{(-n+1)/2} exp[i(\frac{x^2}{4(t+C_2)}$$
$$-\frac{AC_1^2}{n(t+C_2)^n}) + Bt + C_3]$$

Example 5:

With A = 4, B = 7, n = -2 in above definition, we get the following form of a NLS equation:

$$iu_t + x^2(\frac{1}{x^2}u_x)_x + (4|u|^2 + 7)u = 0$$

With the following general solution:

$$u(x,t) = C_1(t+C_2)^{3/2} exp[i(\frac{x^2}{4(t+C_2)}$$
$$+2C_1^2(t+C_2)^2 + 7t + C_3]$$

Definition 6:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and A, k and n are constants in this 1D NLS equation.

$$iu_t + \frac{1}{x^n}(x^n u_x)_x + A|u|^k u = 0$$

With the following solution:

$$u = C_1(t+C_2)^{(-n+1)/2} exp[i[\frac{x^2}{4(t+C_2)} + \frac{2A|C_1|^k]}{2-k-nk}$$
$$*(t+C_2)^{(2-k-nk)/2} + C_3]]$$

Source: Polyanin (2004).

Example 6:

With n = 1, k = 2, A = 3 in above NLS equation in definition, we get the following form of that equation:

$$iu_t + \frac{1}{x}(xu_x)_x + 3|u|^2 u = 0$$

Which yields the following general solution

$$u = C_1 exp[i(\frac{x^2}{4(t+C_2)} - \frac{3|C_1|^2}{t+C_2} + C_3)]$$

Definition 7:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and a and k are constants in this 1D NLS equation.

$$iu_t + u_{xx} + a(1 - e^{-k|u|})u = 0$$

With the following solution:

$$u(x,t) = C_1 exp[i(C_2 x - C_2^2 t + a(1 - e^{-k|C_1|})t + C_3)]$$

Source: R.K. Bullough (1977, 1978).

Example 7:

With the following constant values plugged into above NLS equation in definition

$$a = e^{|C_1|}, k = -2$$

We get the following form of the NLS equation

$$iu_t + u_{xx} + e^{|C_1|}(1 - e^{2|u|})u = 0$$

Which yields the following general solution:

$$u(x,t) = C_1 exp[i(C_2 x - C_2^2 t + e^{|C_1|}(1 - e^{2|C_1|})t + C_3)]$$

Definition 8:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth and f is an arbitrary real function in the 1D NLS equation.

$$iu_t + u_{xx} + f(t, |u|)u = 0$$

With the following general solution:

$$u(x,t) = C_1 exp[i(C_2 x - C_2^2 + \int f(t, |C_1|)dt + C_3)]$$

Source: Polyanin (2004).

Example 8:

With the function f value as the following:

$$f(t, |u|) = te^{|u|}$$

We get the following form of the NLS PDE:

$$iu_t + u_{xx} + te^{|u|}u = 0$$

Which means the traveling wave solutions are the following:

$$u(x,t) = C_1 exp[i(C_2 x - C_2^2 + \int te^{|C_1|}dt + C_3)]$$

$$u(x,t) = C_1 exp[i(C_2 x - C_2^2 + \frac{e^{|C_1|}}{2}t^2 + C_4 + C_3)]$$

Definition 9:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where f and g are arbitrary real functions w.r.t. time t in this1D NLS equation.

$$iu_t + u_{xx} + [f(t)|u|^2 + g(t)]u = 0$$

With the following general solution:

$$u(x,t) = C_1 exp[i(C_2x - C_2^2t) + \int (C_1^2 f(t) + g(t))dt + C_3]$$

Source: Polyanin (2004).

Example 9:

The functions f and g with the following values:

$$f(t) = 2t^2, g(t) = t^3$$

Substituted in the NLS PDE in above definition to give us:

$$iu_t + u_{xx} + [2t^2 |u|^2 + t^3]u = 0$$

As a result, this has the following general solution:

$$u = C_1 exp[i(C_2x - C_2^2t + C_1^2\frac{2}{3}t^3 + \frac{t^4}{4} + C_3 + C_4 + C_5)]$$

Definition 10:

The wave function u(x, t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where f is an arbitrary real function in this 1D NLS equation.

$$iu_t + u_{xx} + f(|u|)u = 0$$

With the following traveling wave solution:

$$u = C_1 exp[i(C_1 exp[i(C_2x - C_2^2t + f(|C_1|)t + C_3)])]$$

Source: Polyanin (2004).

Example 10:

The following function is valued as

$$f(|u|) = cos\,|u|$$

Which gives the following form of above NLS PDE:

$$iu_t + u_{xx} + cos\,|u|\; u = 0$$

So, we have the following traveling wave solutions:

$$u = C_1 exp[i(C_1 exp[i(C_2x - C_2^2t + cos|C_1|\,t + C_3)])]$$

$$u = C_1 exp[i(C_1 exp[i(C_2x - C_2^2t + |C_4|\,t + C_3)])]$$

Definition 11:

The wave function u(x,t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where f is an arbitrary real function and n is a constant in this 1D NLS equation.

$$iu_t + \frac{1}{x^n}(x^n u_x)_x + f(|u|)u = 0$$

With the following general solution:

$$u = C_1 t^{(-n+1)/2} exp[i(\frac{x^2}{4t} + \int f(|C_1|\,t^{(-n+1)/2} dt + C_2)]$$

Source: Polyanin (2004).

Example 11:

With these values in above NLS PDE:

$$f(|u|) = e^{5|u|}, n = -3$$

We get the following form of that NLS PDE:

$$iu_t + x^3(\frac{1}{x^3}u_x)_x + e^{5|u|}u = 0$$

This gives us the following general solutions:

$$u = C_1 t^2 exp[i(\frac{x^2}{4t} + e^{5|C_1|}\frac{t^3}{3} + C_2 + C_3)]$$

Definition 12:

The wave function u(x,t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where f and g are arbitrary functions and a and b are constants in this 1D NLS equation.

$$u_t = (a + ib)u_{xx} + [f(|u|) + ig(|u|)]u$$

With the following traveling wave solution:

$$u = C_1 exp[i(\pm x\sqrt{\frac{f(|C_1|)}{a}} + t[g(|C_1|) - \frac{b}{a}f(|C_1|)] + C_2)]$$

Source: V.S. Berman, Yu. A. Danilov (1981).

Example 12:

The following functions and constants have these values:

$$f(|u|) = e^{|u|}, g(|u|) = ln|u|, a = 2, b = 4$$

Substituted into NLS PDE in above definition, we get:

$$u_t = (2 + 4i)u_{xx} + [e^{|u|} + iln\,|u|]u$$

It has the following traveling wave solution:

$$u = C_1 exp[i(\pm x\sqrt{\frac{e^{|C_1|}}{2}} + t[ln(|C_1|) - 2e^{|C_1|})] + C_2)]$$

Definition 13:

The wave function u(x,t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where f is an arbitrary function and n constant in this 1D NLS equation..

$$iu_t + \frac{1}{x^n}(x^n u_x)_x + f(t, |u|)u = 0$$

With the following general solution:

$$u = C_1 t^{(-n+1)/2} exp[i(\frac{x^2}{4t} + \int f(t, |C_1|\,t^{(-n+1)/2})dt + C_2)]$$

Source: Polyanin (2004).

Example 13:

This function and constant have the following values:

$$f(t, |u|) = t^2 ln\,|u|\,, n = -3$$

Plugging into NLS PDE in above definition, we get

$$iu_t + \frac{1}{x^{-3}}(x^{-3}u_x)_x + t^2 ln\,|u|\,u = 0$$

Which has the following general solution:

$$u = C_1 t^2 exp[i(\frac{x^2}{4t} + \int ln\,|C_1|\,t^4 dt + C_2)]$$

$$u = C_1 t^2 exp[i(\frac{x^2}{4t} + ln\,|C_1|\,\frac{t^5}{5} + C_3 + C_2)]$$

Definition 14:

The wave function u(x,t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where A is a constant in this two dimensional (2D) NLS equation.

$$iu_t + u_{xx} + u_{yy} + A\,|u|^2\,u = 0$$

With the following general solution:

$$u = exp[i(C_2x + C_3y + (AC_1^2 - C_2^2 - C_3^2)t + C_4)]$$

Source: Polyanin (2004), L. Gagnon, P. Winternitz (1988, 1989), N.H. Ibragimov (1995), A.M. Vinogradov (1997).

Example 14:

With A = 5, we get the following NLS PDE:

$$iu_t + u_{xx} + u_{yy} + 5\,|u|^2\,u = 0$$

It has the following general solution:

$$u = exp[i(C_2x + C_3y + (5C_1^2 - C_2^2 - C_3^2)t + C_4)]$$

Definition 15:

The wave function u(x,t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where A and n are constants in this 2D NLS equation.

$$iu_t + u_{xx} + u_{yy} + A\left|u\right|^{2n} u = 0$$

With the following general solution:

$$u = exp[i(C_2x + C_3y + (A\left|C_1\right|^{2n} - C_2^2 - C_3^2)t + C_4)]$$

Source: Polyanin (2004).

Example 15:

If constants A = 6, n = 2, then we get the following form of the above NLS PDE:

$$iu_t + u_{xx} + u_{yy} + 6\left|u\right|^4 u = 0$$

This NLS PDE gives us the following general solution:

$$u = exp[i(C_2x + C_3y + (6\left|C_1\right|^4 - C_2^2 - C_3^2)t + C_4)]$$

Definition 16:

The wave function u(x,t) is assumed complex and is continuous and belongs to a class C^2 of functions; hence this function is differentiable and smooth where the function f is an arbitrary function in this 2D NLS equation.

$$iu_t + u_{xx} + u_{yy} + f(|u|)u = 0$$

With the following traveling wave solution:

$$u = C_1exp[i(C_2x + C_3y + t(f(|C_1|) - C_2^2 - C_3^2) + C_4)]$$

Source: L. Gagnon, P. Winternitz (1988, 1989), N.H. Ibragimov (1995).

Example 16:

The following function added to above NLS PDE, gives this form of that NLS PDE:

$$f(|u|) = e^{2|u|}$$

$$iu_t + u_{xx} + u_{yy} + e^{2|u|}u = 0$$

Which gives us the following traveling wave solution:

$$u = C_1exp[i(C_2x + C_3y + t(2e^{|C_1|} - C_2^2 - C_3^2) + C_4)]$$

II. Conclusion

Nearly all the solutions emitted from the above nonlinear Schrodinger equations, examples are general wave solutions, a few of which are traveling wave solutions. These solutions model waves of light, quanta and such in electromagnetic fields, quantum mechanics. The traveling wave solutions could be utilized in models found in classical mechanics including fluid dynamics, such as water waves.

References

[1] Ablowitz, M. J. and Segur, H., *Solitons and the Inverse Scattering Transform, Society for Industrial and Applied Mathematics*, (SIAM), Philadelphia, 1981.

[2] Akhmediev, N. N. and Ankiewicz, A., *Solitons. Nonlinear Pulses and Beams*, Chapman & Hall, London, 1997.

[3] Faddeev, L. D. and Takhtajan, L. A., *Hamiltonian Methods in the Theory of Solitons,* Springer-Verlag, Berlin, 1987.

[4] Gagnon, L. and Winternitz P., "Lie symmetries of a generalized nonlinear Schrodinger equation. I. The symmetry group and its subgroups," J. Phys. A, Vol. 24, p. 1493, 1988.

[5] Gagnon, L. and Winternitz P., "Lie symmetries of a generalized nonlinear Schrodinger equation. II. Exact solutions," J. Phys. A, Vol. 22, p. 469, 1989.

[6] Novikov, S. P., Manakov, S. V., Pitaevskii, L. B., and Zakharov, V. E., *Theory of Solitons. The Inverse Scattering Method*, Plenum Press, New York, 1984.

[7] Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations* , Chapman & Hall/CRC, Boca Raton, 2004.

[8] Sulem, C. and Sulem, P.-L., *the Nonlinear Schrodinger Equation. Self-Focusing and Wave Collapse*, Springer-Verlag, New York, 1999.

[9] Zakharov, V. E. and Shabat, A. B., "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media," Soviet Physics JETP, Vol. 34, pp. 62–69, 1972.

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