International Journal of Innovation in Science and Mathematics
Volume 6, Issue 2, ISSN (Online): 2347–9051

$\mathcal{J}$- Convergence and $\mathcal{J}$- Core of a Double Sequence in Non-Archimedean Fields

S. Sangeetha
Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai-603 203, India.
Corresponding author email id: sangeethamani79@yahoo.co.in

Date of publication (dd/mm/yyyy): 28/04/2018

Abstract **– The concept of I-Convergence of real bounded sequences was introduced by Kostyrko et al [2, 7, 8] as a generalization of Statistical convergence [4, 5, 9] which is based on the structure of the ideal I of subsets of the set of natural numbers. Later, K. Dems extended the concept of I-convergence of a double sequence in [3, 8, 10]. In this paper, we discussed the idea of I- convergence and I- core of a double sequences in a complete, locally compact, non-trivially valued, non-archimedean field K and also we have proved some basic properties of these concepts, which is analogues to the work presented by Vijaya kumar [15].**

Keywords **– Admissible Ideal, Core of a Double Sequence, Non-archimedean Fields, I- Convergence, I- Core, $\mathcal{K}$-Core, Regular Matrix.**

I. Introduction

Throughout this paper K denotes a complete, locally compact, non-trivially valued, non-archimedean field, where the non-archimedean valuation satisfies the following axioms:

(i) $|x| \geq 0$ *and* $|x| = 0$ *iff* $x = 0$.
(ii) $|xy| = |x||y|$.
(iii) $|x + y| \leq \max\{|x|, |y|\}$ *for all* $x, y \in K$.

For analysis in the classical case a general reference is [6], while for analysis in non-archimedean fields, a general reference is [1, 11].

For a given infinite matrix $A = (a_{pqmn})$, we define

$$Ax = (Ax)_{pq} = \sum_{m=1,n=1}^{\infty,\infty} a_{pqmn} x_{mn}, \qquad p, q = 1,2,3, \ldots,$$

assuming that the series on the right converge and $\{(Ax)_{pq}\}$ is called the A-transform of the double sequence $x = \{x_{mn}\}$.

If, whenever $\{x_{mn}\}$ converges to a limit, $\{(Ax)_{pq}\}$ converges to the same limit, then the matrix $A = (a_{pqmn})$ is said to be regular.

Definition 1.1

Let $C_r(x), r = 1,2,3 \ldots$ be the smallest closed K-convex set containing all points x_{mn} *for* $m, n > r$. We define the core of the double sequence $x = \{x_{mn}\}$ as

$$\mathcal{K}(x) = \bigcap_{r=1}^{\infty} C_r(x).$$

Also define $C_r(u) = \{u' \epsilon K \mid |u' - u| \leq r\}$, where $C(0) = \{u \epsilon K \mid |u| \leq 1\}$.

In this context, we refer to Silvermann-Toeplitz theorem for double sequences and series in non-archimedean fields [12, 13].

Definition 1.2

Let $\{x_{mn}\}$ be a double sequence in K and $x \epsilon$K. We say that $\lim_{m+n\to\infty} x_{mn} = x$ if for each $\varepsilon > 0$, the set $\{(m, n) \epsilon \mathbb{N}^2 : |x - x_{mn}| \geq \varepsilon\}$ is finite. In such a case we say that *x* is the limit of $\{x_{mn}\}$.

Definition 1.3

Let $\{x_{mn}\}$ be a double sequence in K and sϵK. We say that

$$s = \sum_{m=1,n=1}^{\infty,\infty} x_{mn},$$

$$if\ s = \lim_{m+n\to\infty} s_{mn}\ where\ s_{mn} = \sum_{i=1,j=1}^{m,n} x_{ij}, \quad m, n = 1,2, \ldots$$

Remark 1.1

If $\lim_{m+n\to\infty} x_{mn} = x$, then the sequence $\{x_{mn}\}$ is automatically bounded.

Lemma 1.1

$\lim_{m+n\to\infty} x_{mn} = x$ if and only if

(i) $\lim_{n\to\infty} x_{mn} = x, \quad m = 1,2, \ldots$

(ii) $\lim_{m\to\infty} x_{mn} = x, \quad n = 1,2, \ldots$ and

(iii) For each $\varepsilon > 0$ there exists an $N \epsilon \mathbb{N}$ such that $|x - x_{mn}| < \varepsilon$, for all $m, n \geq N$, which we write as $\lim_{m,n\to\infty} x_{mn} = x$.

Lemma 1.2

$\lim_{m,n\to\infty} s_{mn}$ exists if and only if $\lim_{m,n\to\infty} x_{mn} = 0$.

Theorem1.1

In order that whenever a sequence $\{x_{mn}\}$ has a limit x,

$\sum_{m=1,n=1}^{\infty,\infty} a_{pqmn} x_{mn}$, shall converge and $\lim_{p+q\to\infty} \sum_{m=1,n=1}^{\infty,\infty} a_{pqmn} x_{mn} = x$

$i.e., for\ A = (a_{pqmn})$ to be regular it is necessary and sufficient that

$(a)\ \lim_{p+q\to\infty} a_{pqmn} = 0,\ \ m,n = 1,2\ ...,$

$(b)\ \lim_{p+q\to\infty} \sum_{m=1,n=1}^{\infty,\infty} a_{pqmn} = 1$

$(c)\ \lim_{p+q\to\infty} \sup_{m\geq 1} |a_{pqmn}| = 0,\quad n = 1,2, ...,$

$(d)\ \lim_{p+q\to\infty} \sup_{n\geq 1} |a_{pqmn}| = 0,\quad m = 1,2, ...,$

$(e)\ \sup_{p,q,m,n} |a_{pqmn}| < \infty.$

II. I-Convergence of a Double Sequence

Definition 2.1

For any set S ⊂ X (a non-Archimedean normed space over K) a non-empty subset I of a ring R of subsets of S is an ideal in R iff

i. A, B ∈ I implies A ∪ B ∈ I.
ii. A ∈ I, B ∈ R, B ⊂ A implies B ∈ I.

An ideal I is called non-trivial if I ≠ φ and S ∉ I.

Definition 2.2

A non-trivial ideal I is said to be admissible, whenever {x} ∈ I for every x ∈ S.

Definition 2.3

Let I be a non-trivial ideal in N^2. Then a double sequence x = (x_{mn}) is said to be I - convergent to L ∈ K if for every ε > 0 the set $\{(m,n) \in N^2 : |x_{mn} - L| \geq \varepsilon\} \in I$.

Symbolically, it is denoted as

$$I\text{ - }\lim_{m,n\to\infty} x_{mn} = L \quad \text{or} \quad I\text{- } lim\ x = L.$$

For a double sequence x = (x_{mn}) in K,

let $B_x = \{b \in K : \{(m, n) : x_{mn} > b\} \notin I\}$ and

$A_x = \{a \in K : \{(m, n) : x_{mn} > a\} \notin I\}$.

Definition 2.4

Let I be an admissible ideal and x be a sequence in K. Then the I-limit superior of x is given by

$$I - \lim\sup x = \begin{cases} \sup B_x, & if\ B_x \neq \emptyset \\ -\infty, & if\ B_x = \emptyset. \end{cases}$$

Also, the I-limit inferior of x is given by

$$I - \lim\inf x = \begin{cases} \inf A_x, & if\ A_x \neq \emptyset \\ +\infty, & if\ A_x = \emptyset. \end{cases}$$

Theorem 2.1.

If $l_1 = I - lim\ sup\ x$ is finite, then for every positive number ε,

$$\{(m,n): l_1 - \varepsilon < x_{mn} < l_1 + \varepsilon\} \notin I \qquad (1)$$

Conversely, if (1) holds for every positive number ε, then $l_1 = I - lim\ sup\ x$.

Proof. Suppose that $l_1 = I - lim\ sup\ x$ is finite.

Then by definition $B_x \neq \varphi$ implies $l_1 = sup\ B_x$.

Since $l_1 = sup\ B_x$, for a positive number ε there exists b ∈ B_x such that b > $l_1 - \varepsilon$.

Therefore, $\{(m, n) : x_{mn} > b > l_1 - \varepsilon\} \notin I$.

Suppose $\{(m, n) : x_{mn} > l_1 - \varepsilon\} \in I$, then we have $\{(m, n) : x_{mn} > b\} \in I$ (2)
which is not possible as b ∈ B_x. Hence,
$\{(m, n) : x_{mn} < l_1 - \varepsilon\} \notin I$

Next assume $\{(m,n) : x_{mn} > l_1 + \varepsilon\} \notin I$. Then $x_{mn} > l_1 + \varepsilon$ implies $l_1 + \varepsilon \in B_x$ which is a contradiction to the fact that $\beta = sup\ B_x$.

$$\{(m,n) : x_{mn} < l_1 + \varepsilon\} \notin I \qquad (3)$$

Therefore, from (2) and (3),

$\{(m, n) : l_1 - \varepsilon < x_{mn} < l_1 + \varepsilon\} \notin I$ holds.

Converesly, assume (1) holds for every positive number ε.

Since $\{(m,n) : x_{mn} > l_1 - \varepsilon\} \notin I$, we have $B_x \neq \varphi$ so $I - lim\ sup\ x = sup\ B_x$.

If l_1 = sup B_x, it completes the proof.

Suppose $l_1 \neq sup\ B_x$, then there exists b ∈ B_x such that b > l_1.

Taking $\varepsilon = b - l_1 > 0$. That is, $b = l_1 + \varepsilon \in B_x$ implies $\{(m,n) : x_{mn} > l_1 + \varepsilon\} \notin I$ which is a contradiction to (1).

Therefore, l_1 = sup B_x = I − lim sup x.

Hence, the proof is complete.

Similarly, we can have dual statement for I − lim inf x as follows.

Theorem 2.2.

If $l_2 = I - lim\ inf\ x$ is finite, then for every positive number ε,

$$\{(m,n): l_2 - \varepsilon > x_{mn} > l_2 + \varepsilon\} \notin I \qquad (4)$$

Conversely, if (4) holds for every positive number ε, then $l_2 = I - lim\ inf\ x$.

Theorem 2.3.

A double sequence x is I - convergent if and only if $I - \lim\inf x = I - \lim\sup x$.

Proof.

Let $l_1 = I - \lim\sup x$ and $l_2 = I\ lim\ inf\ x$.

Suppose x is I - convergent then $I - lim\ x = L$.

For $\varepsilon > 0$, then $\{(m,n) : |x_{mn} - L| \geq \varepsilon\} \in I$ implies
$\{(m,n): x_{mn} > L + \varepsilon\} \in I$ and
$\{(m,n) : x_{mn} < L - \varepsilon\} \in I$

International Journal of Innovation in Science and Mathematics
Volume 6, Issue 2, ISSN (Online): 2347–9051

Suppose if $\{(m,n):\ |x_{mn} - L| > L + \varepsilon\} \notin I$
then we have $L + \varepsilon \in B_x$.
Also, $l_1 = sup B_x$ implies $L + \varepsilon \leq l_1$.
Therefore, $L < l_1$.
Hence, $\{(m,n):\ |x_{mn} - L| > L + \varepsilon\} \in I$ implies $l_1 \leq L$.
Similarly, we can prove, if
$\{(m,n): x_{mn} < L - \varepsilon\} \in I$,
then $l_2 \geq L$.
Hence,

$$l_1 \leq l_2 \quad (5)$$

But for any sequence x,
$I - lim\ inf\ x \leq I - lim\ sup\ x.$

$$l_2 \leq l_1 \quad (6)$$

From equations (5) and (6), we have $l_1 = l_2$.
Conversely, assume $l_1 = l_2 = L$.
By theorem (2.1),

$$\{(m,n):\ L - \varepsilon/2 < x_{mn} < L + \varepsilon/2\} \notin I.$$

That is, $\{(m,n):\ |x_{mn} - L| < \varepsilon\} \notin I.$
That is, $\{(m,n):\ |x_{mn} - L| \geq \varepsilon\} \in I$
which implies *x* is *I* - convergent to L.
Hence the proof is complete.

Theorem 2.4.

Let I be a non-trivial ideal.

1. If $I - \lim_{m,n\to\infty} x_{mn} = l_1, I - \lim_{m,n\to\infty} y_{mn} = l_2$, then $I - \lim_{m,n\to\infty} (x_{mn} + y_{mn}) = l_1 + l_2$.
2. If $I - \lim_{m,n\to\infty} x_{mn} = l_1, I - \lim_{m,n\to\infty} y_{mn} = l_2$, then $I - \lim_{m,n\to\infty} (x_{mn}\, y_{mn}) = l_1\, l_2$.
3. If I is an admissible ideal, then $\lim_{m,n\to\infty} x_{mn} = l$ implies $I - \lim_{m,n\to\infty} x_{mn} = l$.

Proof.

Given $I - \lim_{m,n\to\infty} x_{mn} = l_1$ and $I - \lim_{m,n\to\infty} y_{mn} = l_2$

Let ε > 0, we have,

$\{(m,n) \in N^2 : |x_{mn} - l_1| \geq \varepsilon\} \in I$ and
$\{(m,n) \in N^2 : |y_{mn} - l_2| \geq \varepsilon\} \in I$.

Then its complement can be represented as,

$\{(m,n) \in N^2 : |x_{mn} - l_1| < \varepsilon\} \notin I$ and
$\{(m,n) \in N^2 : |y_{mn} - l_2| < \varepsilon\} \notin I$.

Consider,

$$\begin{aligned} &|(x_{mn} + y_{mn}) - (l_1 + l_2)| \\ &= |(x_{mn} - l_1) + (y_{mn} - l_2)| \\ &\leq \max\{|x_{mn} - l_1|, |y_{mn} - l_2|\} \\ &< \varepsilon \end{aligned}$$

Therefore,

$$\{(m,n) \in N^2 : |(x_{mn} + y_{mn}) - (l_1 + l_2)| \geq \varepsilon\} \in I$$

Hence, $I - \lim_{m,n\to\infty} (x_{mn} + y_{mn}) = l_1 + l_2$.

2. Consider,

$$\begin{aligned} &|x_{mn}\, y_{mn} - l_1\, l_2| \\ &= |x_{mn} y_{mn} - x_{mn}\, l_2 + x_{mn} l_2 - l_1\, l_2| \\ &= |x_{mn}\,(y_{mn} - l_2) + (x_{mn} - l_1) l_2| \\ &\leq \max\{|x_{mn}|\,|y_{mn} - l_2|, |x_{mn} - l_1||l_2|\} \\ &< \varepsilon \end{aligned}$$

Since (x_{mn}) is bounded there exists an M such that $|x_{mn}| < M$.

And for ε > 0, let us take $|y_{mn} - l_2| < \frac{\varepsilon}{M}$.

Since K is non-trivially valued, there exists $l_2 \in$ K such that $0 < |l_2| < 1$.
Therefore, $|x_{mn} - l_1| < \varepsilon < \frac{\varepsilon}{|l_2|}$.
Hence,

$$|x_{mn}\, y_{mn} - l_1\, l_2| < \max\left\{M.\frac{\varepsilon}{M}, \frac{\varepsilon}{|l_2|}|l_2|\right\} < \varepsilon$$

which implies

$$\{(m,n) \in N^2 : |x_{mn}\, y_{mn} - l_1\, l_2| \geq \varepsilon\} \in I$$

Therefore,

$$I - \lim_{m,n\to\infty} (x_{mn}\, y_{mn}) = l_1\, l_2$$

3. For any ε > 0, $|x_{mn} - l| < \varepsilon$ *(given).*

Since I is non-empty, there exists atleast one (m, n) $\in N^2$ such that $|x_{mn} - l| \geq \varepsilon$ belong to I.

That is, $\{(m,n) \in N^2 : |x_{mn} - l| \geq \varepsilon\} \in I$.

Hence, $I - \lim_{m,n\to\infty} x_{mn} = l$.

This completes the proof.

T

III. I - Core of a Double Sequence

Definition 3.1

Let I be an admissible ideal. For any double sequence $x = (x_{mn}), x_{mn} \in K, \quad m, n = 1, 2, \cdots,$.

Let us define I - core of a double sequence *x* as $I - core(x) = \bigcap_{v \in K} B_x(v)$ for all $v \in K$ where

$$B_x(v) = \{w \in K : |w - v| \leq I - \lim_{m,n\to\infty} sup\, |x_{mn} - v|\}$$

Remark 3.1

In the definition of $\mathcal{K}$ *(x)*, the closed K - convex set $C_n(x)$ contains (x_{mn}) for m, n ∈ K. Hence, in defining the

I−core(x) we have replaced the sequence (x_{mn}) by an arbitrary subsequence in which

$$\{(m,n) \in N^2 : x_{mn} \in D_\varepsilon(a)\} \notin I$$
$$where\ D_\varepsilon(a) = \{x_{mn} \in K : |x_{mn} - a| \le \varepsilon\}.$$

Therefore, I−core(x) $\subset \mathcal{K}$ − core(x), for all x.

Theorem 3.1

Let I be an admissible ideal in N^2 and let
$\{(m,n): |x_{mn} - v| < r + \varepsilon\} \notin I$
$where\ r = I - \lim_{m,n\to\infty} sup\ |x_{mn} - v|$. For any double sequence x = (x_{mn}), if the matrix

$A = (a_{pqmn}), a_{pqmn} \in K$ satisfies the following conditions:

(i) A is regular

(ii) $\lim_{p,q\to\infty} \sup_{m,n\ge 1} |a_{pqmn}| = 1$

then $\mathcal{K} - core(Ax) \subset I - core(x)$.

Proof.

Let x = (x_{mn}) be a double sequence converging to v in K such that A = (a_{pqmn}) is regular and satisfies (ii).

If w is any point of $\mathcal{K}$ (Ax), then

$$|w - v| \le \lim_{p,q\to\infty} \sup_{m,n\ge 1} \left|(Ax)_{pq} - v\right|, v \in K$$

$$= \lim_{p,q\to\infty} \sup_{m,n\ge 1} \left|\sum_{m,n=1}^{\infty,\infty} a_{pqmn} x_{mn} - v\right|$$

$$= \lim_{p,q\to\infty} \sup_{m,n\ge 1} \left|\sum_{m,n=1}^{\infty,\infty} a_{pqmn} x_{mn} - \sum_{m,n=1}^{\infty,\infty} a_{pqmn}\ v\right|$$

{since A is regular}

$$= \lim_{p,q\to\infty} \sup_{m,n\ge 1} \left|\sum_{m,n=1}^{\infty,\infty} a_{pqmn}\ (x_{mn} - v)\right|$$

$$= \lim_{p,q\to\infty} \sup_{m,n\ge 1} \left|\sum_{m,n=1}^{\infty,\infty} a_{pqmn}\right| \left|\sum_{m,n=1}^{\infty,\infty} x_{mn} - v\right|$$

Given $r = I - \lim_{m,n\to\infty} sup\ |x_{mn} - v|$ and let

$$E = \{(m,n) \in \mathrm{N}^2 : |x_{mn} - v| < r + \varepsilon\}$$

Then $E \notin I$ and we have

$$\lim_{p,q\to\infty} \sup_{m,n\ge 1} |a_{pqmn}||x_{mn} - v| \le |x_{mn} - v|$$

$$< r + \varepsilon$$

$$\le I - \lim_{p,q\to\infty} \sup_{m,n\ge 1} |x_{mn} - v|$$

Therefore, $|w - v| \le I - \lim_{p,q\to\infty} \sup_{m,n\ge 1} |x_{mn} - v|$ for

$E \notin I$ implies $w \in I - core(x)$.

Hence, $\mathcal{K} - core(Ax) \subset I - core(x)$.

III. Conclusion

Many authors discussed the concept of I- convergence and I- Cauchy sequence for real bounded sequences and double sequences in classical case. In this paper, we have proved some basic properties of these concepts in a complete, locally compact, non-trivially valued, non-archimedean field K.

References

[1] G. Backman, Introduction to p-adic numbers and valuation theory, Academic Press, 1964.

[2] P. Das, P. Kostyrko, "I and I∗ - G. Backman, Introduction to p-adic numbers and valuation theory, Academic Press, 1964. convergence of double sequences", *Mathematica slovaca*, 58(5), 2008, pp. 605- 620.

[3] V. Dems, "On I-Cauchy sequences", *Real Analysis Exchange*, 30(1), 2005, pp. 123-128.

[4] J. A. Fridy, "On Statistical Convergence", *Analysis*, 5, 1985, pp. 301-313.

[5] J.A. Fridy, C. Orhan, "Statistical limit superior and limit inferior", *Proc.Amer. Math. Soc*., 125, 1997, pp. 3625-3631.

[6] G.H. Hardy, *Divergent Series*, Oxford, 1949.

[7] P. Kostyrko, T. Salat, W. Wilczynski, "I-Convergence", *Real Analysis Ex-change*., 26(2), 2001, pp. 669-686.

[8] B. K. Lahiri, P. Das, "Further results on I-limit superior and I-limit inferior", *Mathematical Communications*, 8, 2003, pp. 151-156.

[9] M. Mursaleen, Osama H .H. Edely, "Statistical Convergence of double sequences", *J. Math. Anal. Appl*., 288, 2003, pp. 223-231

[10] A. Nabiev, S. Pehlivan, M. Gurdal, "On I-cauchy sequences", *Taiwanese Journal of Mathematics*, 11(2), 2007, pp. 569-576.

[11] L. Narici, E. Beckenstein, G. Bachman, *Functional Analysis and Valuation Theory*, Marcel Dekker, New York, 1971.

[12] P.N. Natarajan, V. Srinivasan, "Silvermann - Toeplitz theorem for double sequences and series and its application to Norlund means in non- archimedean fields", *Annales Mathematiques Blaise Pascal*, 9, 2002, pp. 85-100.

[13] G. M. Robison, "Divergent double sequence and series", *Trans. Amer. Math. Soc*., 28, 1926, pp. 50-73.

[14] S. Sangeetha, V. Srinivasan, "I-Cauchy sequence and I-core of a sequence in Ultrametric Fields", *Int. J. of Pure and Applied Mathematics*, 113(10), 2007, pp. 19-17.

[15] Vijay Kumar, "I-core of double sequences", *Int. J. Contemp. Math. Sciences,* 2, 2007, pp. 1137-1145.