A Collective Risk Theory in Reinsurance

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Abstract — This paper presents the optimal reinsurance based on an assessment of the reinsurance arrangements for a large life insurer. The main aim is to compute the reinsurance structure, based on actual insurer data using a modified mean-variance criteria that maximizes the retained premiums and minimizes the variance of retained claims, while keeping the retained risk exposure constant, assuming a given level of risk appetite. The portfolio of life and disability policies uses quota-share, surplus and a combination of both the quota-share and the surplus reinsurance. Alternative reinsurance arrangements are compared using the modified mean-variance criteria to assess the optimal reinsurance strategy. The analysis takes into the account recent claims experience as well as the actual premiums paid by the insured lives and to the reinsurers. The optimal reinsurance cover depends on many factors including retention levels, premiums and the variance of sum insured values (and therefore claims), as a result an insurer should assess the trade-off between the retained premiums and the variance of retained claims based on its own experience and risk appetite.

Keywords — Cede, Portfolio, Treaty Reinsurance, Reinsurer, Proportional Reinsurance.

I. INTRODUCTION

One of the chief risk and capital management tools available to the insurers is Reinsurance [1-8]. Its significance is however, hardly the known outside the insurance sector. Reinsurance is insurance for the insurers. Insurers buy reinsurance for the risks they cannot or do not wish to retain fully themselves. reinsurers help the industry to provide protection for a wide range of risks, including the largest and most complex risks →including underwriting, asset management and capital management [9-14]. Reinsurance also makes insurance less expensive and more secure, thus benefiting the policyholders who get more protection at a lower cost. reinsurers apply the sophisticated risk management processes, including the risk monitoring and the risk modeling to ensure that the promise to pay future possible claims is honored [15-16]. In this paper, proposes the various issues relating to reinsurance; including its concept, benefits, applications, methods, functions, forms, types, and how reinsurers deal with risk. The remarks of it conclude with a snapshot of how reinsurance applies to life insurers.

II. NEED FOR INSURANCE

A Direct insurer needs reinsurance for the following reasons:

- limit (as much as possible) annual fluctuations in the losses he must bear on his own account; and
- To be protected in case of catastrophe.

In many cases the premiums to the insurance company are not enough to carry the risk. This is the case of large claims. In those cases the insurer shares a part of the risk with other companies. Sharing the risk as well as the premiums is done by reinsurance contracts, which are mutual agreements between insurance companies. Sometimes the insurance companies have agreements about the reinsuring certain parts of the portfolios. In this case, consider the reinsurance that applies to the individual claims. If the claim Size is x the insurer retains a part h(x); Where, h: $R_+ \rightarrow R_+$ is an increasing function, such that h(0) = 0 and $0 \leq h(x) \leq x$ for all $x \geq 0$: The reinsurance covers the remain part x - h(x): We assume that reinsurance premiums are payable continuously and that the reinsurance pays its share of a claim as soon as that claim occurs. The function h(x) determines the rule of reinsurance. The aggregate sum of claims for the insurer is equal to $T_i = \sum_{i=1}^{N_i} h(X_i)$ The sum of claims for reinsurer is $T_{i}^R = T_i - T_i^I$.

A. Proportional Reinsurance

In all varieties of proportional reinsurance, the direct insurer and the reinsurer divide premiums and losses (claims) between them at a contractually defined ratio. According to the type of treaty, this ratio may be the same for all risks covered by the contract (quota share reinsurance), or it may vary from risk to risk (all other proportional reinsurance types). In all cases, however, the reinsurer’s share of the premiums is directly proportional to his obligation to pay any losses. For example, if the reinsurer accepts 75% of a particular risk and the direct insurer retains 25%, the premium is apportioned at a ratio of 75:25.

Suppose the insurer chooses proportional reinsurance with retention level b ∈ [0; 1]: In this case, the function h(x) = bx: The premium rate for the reinsurance is given by $(1 + \eta)(1 - b)\lambda\mu$, where $\eta > 0$ is the relative safety loading, defined by the reinsurance company.

We consider the case $\eta > 0$: The premium rate for the insurer is:

$\lambda\mu[(1+\theta)-(1+\eta)](1-b)] = \lambda\mu [ b(1+\eta)-(\eta-\theta)]$

$V(t,b) = v + \lambda\mu[b(1+\eta)-(\eta-\theta)] t - \sum_{k=1}^{N_i} bX_k$

In order that the net profit condition is fulfilled we need

$\frac{\lambda\mu[b(1+\eta)-(\eta-\theta)]}{\lambda\mu b} > 1$
\[ b > 1 - \frac{\theta}{\eta} \]

Let \( M_{k}(r) \) be the moment generating function of the individual claim amount distribution evaluated at \( r \). Then the adjustment coefficient \( R(b) \) under proportional reinsurance is the unique positive solution of the equation

\[
\lambda \mu [M_{k}(d) - 1] - \lambda \mu [b(1 + \eta) - (\eta - \theta)]= 0 \quad (2)
\]

Let \( \psi(\nu; b) \) denote the probability of ultimate ruin when the proportional reinsurance is chosen. Then

\[
\psi(\nu; b) = P[V(t, b) < 0 \text{ for some } t > 0]
\]

Our objective is to find the retention level that minimizes \( \psi(\nu; b) \): According to the Lundberg inequality, the retention level will be optimal, if the corresponding Lundberg exponent \( R \) is maximal. It is known that there is a unique \( b \in [0; 1] \) where the maximum is attained. If the maximize \( b > 1 \); it follows from the uni-modality that the optimal \( b \) is \( 1, \) i.e. no reinsurance is chosen.

The next result gives the optimal retention level \( b \) and maximal adjustment coefficient \( R(b) \).

**Lemma 1.1:** The solution of equation (2) is given by

\[
R(b) = \frac{(1 + \eta)\mu - [1 - M_{k}(r)]}{(\eta - \theta)\mu} \quad (3)
\]

where \( b \rightarrow r(b) \) is invertible.

**Proof:** Assume that \( r(b) = br(b) \); where \( R(b) \) will be the maximal value of the adjustment coefficient and \( r(b) \) is invertible. If we consider the function \( \tau \rightarrow b(\tau) \); it follows that

\[
b(\tau) = \frac{(\eta - \theta)\mu r}{(1 + \eta)\mu r - [1 - M_{k}(\tau)]} \quad (4)
\]

Now \( R(b(\tau)) = \frac{r}{b(\tau)} \) in details given by (3)

**Theorem 1.1:** Assume that \( M_{k}(r) < 1 \): Suppose there is a unique solution \( \tau \) to \( M_{k}'(\tau) - (1 + \eta)\mu = 0 \)

Then \( \tau > 0 \); the maximal value of \( R(b(\tau)) \) and the retention level \( b(\tau) \) are given by (3) and (4).

**Proof:**

The necessary condition for maximizing the value of the adjustment coefficient is given by equation (5).

Since \( \dot{R}(b(0)) = \frac{1}{\eta - \theta} > 0 \), the function \( R(b(r)) \) is strictly increasing at 0.

The second derivative in zero

\[
\dot{R}'(b(0)) = -\frac{1}{(\eta - \theta)\mu} EX^2 < 0
\]

The above equation shows that \( R(b(r)) \) is strictly concave. Consequently, the function \( R(b(r)) \) has an unique maximum in \( \tau \), which is the solution of (3). The retention level is given by (4).

**Remark 1.1** Note that the value of the adjustment coefficient does not depend on \( c \) but on the relative safety loadings only.

**B. Non-Proportional Reinsurance**

Excess - of - Loss Reinsurance (XL): Non-proportional reinsurance is structured like a conventional insurance policy; the reinsurer pays all or a predetermined percentage of the claims which fall between a defined lower and upper cover limit. For the parts of claims below of above limits, the primary insurer has to carry the risk on its own or it may reinsure it under other contracts.

Different to proportional reinsurance, the premium is set independently of the original business. Non-proportional reinsurance arrangements are characterized by a distribution of liability between the cedent and the reinsurer on the basis of losses rather than sums insured, as in case of proportional arrangements. In fact, no insurance amount is ceded under excess of loss treaties; what is ceded is losses and premiums. As compensation for the cover granted, the reinsurer receives part of the original premiums and not part of the premium corresponding to the sum reinsured as in proportional reinsurance.

The following common characteristics differentiate them from proportional treaties.

1. The size of cession is not determined case by case.
2. Administrative costs are substantially reduced.
3. Usually there is no profit commission.
4. Reinsurance premium is worked out on the basis of exposure and past loss experience.

The Excess - of - Loss reinsurance is non-proportional type of reinsurance. The insurer covers each individual claim up to a certain retention level \( M \); i.e. when a claim of size \( X \) occurs, the insurer pays \( X_{R} = \min(X; M) = h(x) \) and the reinsurer \( X_{R} = X - X_{R} = \max(X - M; 0) = (X - M)_{+} \), so that \( X = X_{R} + X_{R} \); Suppose the number of claims \( N(t) \) follows an ordinary renewal process. Hence the insurer risk process at time \( t \) is

\[
Y_{M}(t) = (c - c_{M})t - \sum_{i=1}^{N(t)} \min(Y_{i}, M)
\]

Where, \( c_{M} \) is the XL reinsurance premium. For a given \( M \), the adjustment coefficient \( R_{M} \) is the unique positive root of \( g_{M}(t) = 1 \);

If it exists with

\[
g_{M}(r) = E[e^{rX_{M}}]E[e^{-(c-c_{M})r}] = R_{M} \]

**C. Stop-Loss Reinsurance**

In general, aggregate excess of loss or stop-loss reinsurance is a form of reinsurance coverage for an insurance company that deals with the accumulation of the reinsurer’s net loss position over a specified period of time, usually one calendar year. As is true with any form of excess of loss reinsurance, the reinsurance agreement includes loss retention (for the reinsurer) and a limit (for the reinsurer) of reinsurance. One unique feature of aggregate or stop-loss reinsurance is that it deals with the net loss experience of the reinsurer.

The Stop - Loss reinsurance works similarly to the XL reinsurance, but it covers the total amount of claims. For stop - loss contract with retention level (deductible) \( d \), the amount paid by reinsurer to the insurer is

\[
I_{d} = \begin{cases} 
0 & \text{if } S \leq d \\
S - d & \text{if } S > d
\end{cases}
\]

Sometimes: \( I_{d} = (S - d)^{+} \); Note that \( I_{d} \) as a function of the aggregate claims \( S \) is also a random variable. The amount of claims retained by the insurer is

\[
\min(S, d) = S - I_{d} = \begin{cases} 
S & \text{if } S \leq d \\
d & \text{if } S > d
\end{cases}
\]

Thus, the amount retained is bounded by \( d \), which explains the name stop - loss contract.

The expected claims paid by the reinsurer is given by

\[
E[(c - c_{M})I_{d}] = c - c_{M} \int_{0}^{\infty} \int_{0}^{\min(S, d)} e^{-rS} f_{S}(s) f_{I_{d}}(d) dS dr
\]

Where, \( f_{S}(s) \) is the probability density function of the aggregate claims \( S \) and \( f_{I_{d}}(d) \) is the probability density function of the retention level \( I_{d} \).
\( EI_d = \int_d^\infty (x - d) f_s(x) dx \)

\( = ES - d + \int_d^\infty (d - x) f_s(x) dx \)  

\( = \int_d^\infty (1 - F_s(x)) \)  

\( = ES - \int_0^d (1 - F_s(x)) dx \)  

When ES is available, (7) and (9) are preferable by numerical integration. The (8) and (9) hold for general distribution, including discrete and mixed. If the distribution is given, (6) is the most tractable formula.

III. CONCLUSIONS

This paper proposed to call a reinsurance arrangement optimal if it minimizes the total capital of the primary insurer and the reinsurer. This optimal reinsurance produces the cheapest price for primary insurance policies, so is an attracting equilibrium under market competition. An interesting relationship is observed between the total capital and the tail correlation between the losses of the insurer and the reinsurer. A multivariate normal model and a numerical example are analyzed to get some insight into the nature of an optimal treaty. This paper fills a gap in the existing literature on optimal reinsurance, in which the capital cost of the reinsurer has not been adequately addressed. Our approach establishes a close link between reinsurance and pricing of insurance and reinsurance policies. In a competitive market, reinsurance not only provides the ceding insurer a tool of risk transfer, but also satisfies the reinsurer with a fair amount of profit, and benefits primary policyholders by reducing the cost of insurance.

REFERENCES


