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On the Complete Elliptic Integrals and Babylonian Identity X: Other Approximation for the Complete Elliptic Integral of the Second Kind

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Abstract **– Using the Theorem 3 of previous paper, we evaluate the complete elliptic integral of the second kind in an approximately equal analytical closed form, by means of Bessel functions.**

I. Introduction

By means of the Theorem 3 of previous paper [1], we proved the following approximately equal analytical closed form:

$$E(k) \cong \frac{\pi}{2} I_0(2k) + \frac{\pi(10k^8 + 112k^6 + 630k^4 + 1260k^2 - 315k\sinh(2k) - 315\cosh(2k) + 315)}{630} - \frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912},$$

for $0 < k \leq 1/2$.

II. Theorem

Theorem 1. *For* $0 < k \leq 1/2$, *then*

$$E(k) \cong \frac{\pi}{2} I_0(2k) + \frac{\pi(10k^8 + 112k^6 + 630k^4 + 1260k^2 - 315k\sinh(2k) - 315\cosh(2k) + 315)}{630} - \frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912},$$

where $E(k)$ denotes the complete elliptic integral of second kind and $I_v(z)$ denotes the modified Bessel function of the first kind.

Proof. We will prove the equality above, developing the left hand side of the equation below

$$(1) E(k) - \frac{\pi}{2} I_0(2k) = I_k.$$

First, in [1], Theorem 3, we proved the identity

$$(2)\ E(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n} \frac{k^{2n}}{(n!)^2},$$

On the other hand, we calculated

$$(3)\ \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2} k^{2n} = \frac{\pi}{2} I_0(2k),$$

we subtract (3) of (2), and obtain

$$(5)\ E(k) - \frac{\pi}{2} I_0(2k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n} \frac{k^{2n}}{(n!)^2} - \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2} k^{2n} = \sqrt{\pi} \sum_{n=0}^{\infty} \left[\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n - 1\right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2} k^{2n}$$

$$= \sqrt{\pi} \sum_{n=0}^{4} \left[\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n - 1\right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2} k^{2n} + \sqrt{\pi} \sum_{n=5}^{\infty} \left[\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n - 1\right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2} k^{2n}$$

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$$= -\frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912} + \sqrt{\pi}\sum_{n=5}^{\infty}\left[\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n - 1\right]\frac{\Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2}k^{2n}.$$

we note that, if $n \geq 5$, then

$$(6)\ \left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n - 1 < [(n+1)! + 1]\frac{\left(-\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n} < (n+1)!\frac{\left(-\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n} < -\frac{(n+1)!}{\left(\frac{1}{2}\right)_n}.$$

we take (6) into (5)

$$(7)\ E(k) - \frac{\pi}{2}I_0(2k) < -\frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912} - \sqrt{\pi}\sum_{n=5}^{\infty}\left[\frac{(n+1)!}{\left(\frac{1}{2}\right)_n}\right]\frac{\Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n (n!)^2}k^{2n}$$

$$= -\frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912} - \sqrt{\pi}\sum_{n=5}^{\infty}\frac{(n+1)!\,\Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n (n!)^2}k^{2n}$$

we calculated

$$(8)\ \sqrt{\pi}\sum_{n=5}^{\infty}\frac{(n+1)!\,\Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n (n!)^2}k^{2n} = -\frac{1}{630}\pi(10k^8 + 112k^6 + 630k^4 + 1260k^2 - 315k\sinh(2k) - 315\cosh(2k) + 315).$$

From (7) and (8), I obtain

$$E(k) - \frac{\pi}{2}I_0(2k) < \frac{\pi(10k^8 + 112k^6 + 630k^4 + 1260k^2 - 315k\sinh(2k) - 315\cosh(2k) + 315)}{630} - \frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912},$$

hence,

$$E(k) < \frac{\pi}{2}I_0(2k) + \frac{\pi(10k^8 + 112k^6 + 630k^4 + 1260k^2 - 315k\sinh(2k) - 315\cosh(2k) + 315)}{630} - \frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912},$$

since $0 < k < 1/2$,we conclude easily that

$$E(k) \cong \frac{\pi}{2}I_0(2k) + \frac{\pi(10k^8 + 112k^6 + 630k^4 + 1260k^2 - 315k\sinh(2k) - 315\cosh(2k) + 315)}{630} - \frac{\pi k^2(1831k^6 + 6976k^4 + 43776k^2 + 184320)}{294912}.$$

This finished the proof. The reader may see the Table 1.

REFERENCES

[1] http://www.bmsa.us/admin/uploads/NYLcGa.pdf
[2] http://www.bmsa.us/admin/uploads/Mn95aZ.pdf
[3] http://www.bmsa.us/admin/uploads/W9A5nt.pdf
[4] http://www.bmsa.us/admin/uploads/n8VdcA.pdf
[5] http://www.bmsa.us/admin/uploads/sI2hXE.pdf

Table 1: In this table, I have: first column: m; second column: $k = 1/m$; third column: $E\left(\frac{1}{m}\right)$; fourth column: G_1^*; fifth column: $\frac{G_1^*}{E\left(\frac{1}{m}\right)}$, where $G_1^* = \frac{\pi}{2} I_0\left(\frac{2}{m}\right) + \frac{\pi}{630}\left[\frac{10}{m^8} + \frac{112}{m^6} + \frac{630}{m^4} + \frac{1260}{m^2} - \frac{315}{m}\sinh\left(\frac{2}{m}\right) - 315\cosh\left(\frac{2}{m}\right) + 315\right] - \frac{\pi}{294912\, m^2}\left(\frac{1831}{m^6} + \frac{6976}{m^4} + \frac{43776}{m^2} + 184320\right)$.

2	$\frac{1}{2}$	1.4674622093394271554	1.467472193234227089	1.0000068035106705935
3	$\frac{1}{3}$	1.5262092342121874283	1.5262093847388007688	1.0000000986277700109
4	$\frac{1}{4}$	1.5459572561054650349	1.5459572642020216512	1.0000000052372448101
5	$\frac{1}{5}$	1.5549685462425292834	1.5549685470940266464	1.0000000005475978051
6	$\frac{1}{6}$	1.5598305371286827833	1.5598305372646783102	1.00000000008718609084
7	$\frac{1}{7}$	1.5627511292627544183	1.5627511292916713235	1.00000000001850384534
8	$\frac{1}{8}$	1.5646423092625568944	1.5646423092701313908	1.00000000000484104028
9	$\frac{1}{9}$	1.5659369093229590304	1.5659369093252846767	1.0000000000014851469
10	$\frac{1}{10}$	1.5668619420216682912	1.5668619420224774825	1.00000000000051644075
11	$\frac{1}{11}$	1.5675458333404971662	1.5675458333408086578	1.00000000000019871291
12	$\frac{1}{12}$	1.568065688643469198	1.5680656886435995283	1.00000000000008311535
13	$\frac{1}{13}$	1.568470079184143107	1.5684700791842015892	1.00000000000003728614
14	$\frac{1}{14}$	1.5687908392881918108	1.568790839288219663	1.00000000000001775391
15	$\frac{1}{15}$	1.5690495404018847541	1.5690495404018987169	1.00000000000000889885
16	$\frac{1}{16}$	1.5692612206533624497	1.569261220653369769	1.00000000000000466422
17	$\frac{1}{17}$	1.5694366235819572228	1.5694366235819612131	1.00000000000000254254
18	$\frac{1}{18}$	1.5695835902798978113	1.5695835902799000636	1.00000000000000143498
19	$\frac{1}{19}$	1.5697079520534112774	1.5697079520534125887	1.00000000000000083537
20	$\frac{1}{20}$	1.5698141184163880256	1.5698141184163888105	1.00000000000000050001
21	$\frac{1}{21}$	1.5699054737256423517	1.5699054737256428335	1.00000000000000030688
22	$\frac{1}{22}$	1.5699846504660967064	1.569984650466097009	1.00000000000000019268
23	$\frac{1}{23}$	1.5700537211604400414	1.5700537211604402353	1.00000000000000012351
24	$\frac{1}{24}$	1.5701143354650531983	1.570114335465053325	1.00000000000000008069
25	$\frac{1}{25}$	1.5701678196428456723	1.5701678196428457565	1.00000000000000005363
26	$\frac{1}{26}$	1.5702152497673788669	1.5702152497673789238	1.00000000000000003623
27	$\frac{1}{27}$	1.570257506297885084	1.570257506297885123	1.00000000000000002484
28	$\frac{1}{28}$	1.5702953152530112401	1.5702953152530112672	1.00000000000000001726
29	$\frac{1}{29}$	1.5703292796166320983	1.5703292796166321173	1.00000000000000001215
30	$\frac{1}{30}$	1.5703599035372227177	1.5703599035372227313	1.00000000000000000866
31	$\frac{1}{31}$	1.5703876111506035821	1.5703876111506035919	1.00000000000000000624
32	$\frac{1}{32}$	1.5704127613492695165	1.5704127613492695236	1.00000000000000000454