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On the Generating Functions of Numerical Sequences

Sergio Falcon
Department of Mathematics, University of Las Palmas de Gran Canaria, 35004 Las Palmas, Spain.
Corresponding author email id: sergio.falcon@ulpgc.es

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Abstract **– In this paper we study the generating function of an integer sequence and its relation with the recurrence relation between the terms of the sequence. We recall some generating functions previously studied and create new ones especially from the convolution of sequences and the derivatives of the initial generating functions. With respect to the sum and the convolution of integer sequences, an abelian field is found. Finally, we study the inverse sequence with respect to the convolution.**

Keywords **– k-Fibonacci Numbers, k-Fibonomial Numbers, Generating Function, Recurrence Relation, Convolution, Abelian Field, MSC2000 - 15A36, 11C20, 11B39.**

I. Introduction

k- Fibonacci numbers [2, 3] are defined as the sequence $\{F_{k,n}\}_{n\in N}$ such that $F_{k,0} = 0$, $F_{k,1} = 1$ and $F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}$ for $n \geq 1$.

In [1], the k-Fibonacci factorial numbers $n_{F_k}!$ are introduced. It is the product of the k-Fibonacci numbers from $F_{k,n}$, $F_{k,n-1}$ to $F_{k,1}=1$. That is: $n_{F_k}! = F_{k,n}\cdot F_{k,n-1}\cdots F_{k,1}$

1.1. *Definition*

k-Fibonomial numbers are defined in the following form: $\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = \dfrac{F_{k,n}\cdot F_{k,n-1}\cdots F_{k,n-j+1}}{F_{k,j}\cdot F_{k,j-1}\cdots F_{k,2}\,F_{k,1}} = \dfrac{n_{F_k}!}{j_{F_k}!\cdot(n-j)_{F_k}!}$

where n and j are non-negative integers and $0 \leq j \leq n$.

We will suppose that $\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = 0,\ \forall j > n.$

1.2. *Some Properties of the k-Fibonomial Numbers*

In the sequel we present some properties of the k-Fibonomial numbers:

- $\begin{bmatrix} n \\ 0 \end{bmatrix}_{F_k} = \begin{bmatrix} n \\ n \end{bmatrix}_{F_k} = 1$ and $\begin{bmatrix} n \\ 1 \end{bmatrix}_{F_k} = \begin{bmatrix} n \\ n-1 \end{bmatrix}_{F_k} = F_{k,n}$
- Symmetry property: $\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = \begin{bmatrix} n \\ n-j \end{bmatrix}_{F_k}$.
- $F_{k,n-j}\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = F_{k,n}\begin{bmatrix} n-1 \\ j \end{bmatrix}_{F_k}$ from where $\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = \dfrac{F_{k,n}}{F_{k,n-j}}\begin{bmatrix} n-1 \\ j \end{bmatrix}_{F_k}$
- A similar formula is $F_{k,j}\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = F_{k,n}\begin{bmatrix} n-j \\ j-1 \end{bmatrix}_{F_k}$

International Journal of Innovation in Science and Mathematics
Volume 9, Issue 4, ISSN (Online): 2347–9051

- Addition formula: $\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = F_{k,j-1}\begin{bmatrix} n-1 \\ j \end{bmatrix}_{F_k} + F_{k,n-j+1}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{F_k}$

1.3. *On the Powers of the K-Fibonacci Numbers*

Main theorem proven in [1] was the following: For r ≥ 1, $F_{k,n}^r$ is a combination of the powers $F_{k,n-j}^r$ for j = 1, 2,…r of the form $F_{k,n}^r = \sum_{j=1}^{r+1}(-1)^{\left\lfloor \frac{j-1}{2} \right\rfloor}\begin{bmatrix} r+1 \\ j \end{bmatrix}_{F_k} F_{k,n-j}^r$

For instance:

r = 1: $F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}$

r = 2: $F_{k,n}^2 = (k^2+1)F_{k,n-1}^2 + (k^2+1)F_{k,n-2}^2 - F_{k,n-3}^2$ (1)

r = 3: $F_{k,n}^3 = (k^3+2k)F_{k,n-1}^3 + (k^4+3k^2+2)F_{k,n-2}^3 - (k^3+2k)F_{k,n-3}^3 - F_{k,n-4}^3$

II. Generating Functions for the Powers of the K-Fibonacci Numbers

f(x) is the ordinary generating function of the numerical sequence $a_n \;{}_{n\in N}$ if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, [4, 5, 6]. We will indicate that as $f(x) \mapsto a_n \;{}_{n\in N}$.

In [1], the following formulas are proven, being $gf\left[F_{k,n}^r\right]$ the generating function of the r-th power of the k-Fibonacci numbers (without $F_{k,0} = 0$:

$$gf\left[F_{k,n}\right] = \frac{1}{1 - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{F_k} x - x^2} = \frac{1}{1 - k\,x - x^2} \tag{2}$$

$$gf\left[F_{k,n}^2\right] = \frac{1-x}{1 - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{F_k} x - \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{F_k} x^2 + x^3} = \frac{1-x}{1-(k^2+1)(x+x^2)+x^3} \tag{3}$$

$$gf\left[F_{k,n}^3\right] = \frac{1-2k\,x-x^2}{1 - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{F_k} x - \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{F_k} x^2 + \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{F_k} x^3 + x^4} = \frac{1-2k\,x-x^2}{1-(k^3+2k)x-(k^4+3k^2+2)x^2+(k^3+2k)x^3+x^4}$$

2.1. *Sequences that Start with Zeros*

Given a generating function of the form $\frac{N(x)}{P(x)}$ where P(x) is a polynomial, if the sequence begins with "m" zeros, we must multiply the generating function by x^m.

Example 1

$f(x) = \frac{1}{1-2x-3x^2}$ is the generating function of the sequence {1, 2, 7, 20, 61, 182, …}. Then, $g(x) = \frac{x^3}{1-2x-3x^2}$ is the generating function of the sequence {0, 0, 0, 1, 2, 7, 20, 61, 182,…}

International Journal of Innovation in Science and Mathematics
Volume 9, Issue 4, ISSN (Online): 2347–9051

Let $U = \ U_n \ = \ U_0, U_1, U_2, \ldots$ be a numerical sequence. We do the sequence $\{V_n\}$ such that $V_0 = 0$, $V_i = U_{i-1}$ for $i \geq 1$. Then $\{V_n\} = \{0, U_0, U_1, U_2, \ldots\}$. And the sequence $\{W_n\}$ such that $W_0 = 0$, $W_1 = 0$ and $W_i = U_{i-2}$, that is $\{W_n\} = \{0, 0, U_0, U_1, U_2, \ldots\}$. Let us suppose $\{Z_n\} = \{U_n\} + a\{V_n\} + b\{W_n\} = \{U_0, U_1 + aU_0, U_2 + aU_1 + bU_0, \ldots\}$. If $f(x) \mapsto \ U_n$, then $x f(x) \mapsto \ V_n$ and $x^2 f(x) \mapsto \ W_n$. So, $f(x) \ 1+ax+bx^2 \ \mapsto \ Z_n$

Example 2

In this example we will apply this formula. $f(x) = \dfrac{1}{1-x-x^2-x^3} \mapsto \ 1, 1, 2, 4, 7, 13, 24, 44, \ldots$

$$x f(x) = \frac{x}{1-x-x^2-x^3} \mapsto \ 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$$

$$x^2 f(x) = \frac{x^2}{1-x-x^2-x^3} \mapsto \ 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$$

Then: $f(x) \ 1+3x-2x^2 \ = \dfrac{1+3x-2x^2}{1-x-x^2-x^3}$ generates the sequence $1, 1, 2, 4, 7, 13, 24, 44, \ldots \ +3 \ 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$ $-2 \ 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$, that is $1, 4, 3, 8, 15, 26, 49, 90, \ldots$

Finally, if $f(x) = \dfrac{N(x)}{P(x)}$ is an ordinary generating function, necessarily the constant term of P(x) is not null.

2.2. *Some Generating Functions*

1. The generating function of the sequence $\{a^n\}$ is $f(x) = \dfrac{1}{1-ax}$ because $\dfrac{1}{1-ax} = \ 1-ax \ ^{-1} = \sum_{j=0}^{\infty} (-1)^j \dbinom{-1}{j} (ax)^j$

   $= \sum_{j=0}^{\infty} (-1)^j \ -1 \ ^j a^j x^j$ that is

$$\frac{1}{1-ax} \mapsto \ a^n \tag{4}$$

In particular, $\dfrac{1}{1-x} \mapsto \ 1$

2. The generating function of the classical Fibonacci numbers is

$$\frac{x}{1-x-x^2} \mapsto \ F_n \ _{n \in N} \tag{5}$$

And for the Pell numbers

$$\frac{x}{1-2x-x^2} \mapsto \ P_n \ _{n \in N} \tag{6}$$

3. The generating function of the Catalan numbers is $\dfrac{1-\sqrt{4x}}{2x} \mapsto \ 1, 1, 2, 5, 14, 42, 132, \ldots$

2.3. *Theorem 1*

If f(x) is the generating function of the sequence $\ a_n \ _{n \in N}$, then $\dfrac{f(x)}{1-x}$ is the generating function of the sequence of partial sums $\left\{ \sum_{j=0}^{n} a_j \right\}$

Proof.

$$f(x) \ = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

$$x\, f(x) \ = \ a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n + \cdots$$

$$x^2 f(x) = a_0 x^2 + a_1 x^3 + \cdots + a_{n-2} x^n + \cdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$(1+x+x^2+\cdots) f(x) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j\right)x^n \to \frac{1}{1-x} f(x) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j\right)x^n \to \frac{f(x)}{1-x} \mapsto \left\{\left(\sum_{j=0}^{n} a_j\right)\right\}_{n\in N}$$

Example 3

From Formula (6), $\frac{x}{1-2x-x^2} \mapsto 0, 1, 2, 5, 12, 29, 70, \cdots$ (Pell numbers). Then $\frac{x}{1-x \ \ 1-2x-x^2} \mapsto 0, 1, 3, 8, 20, 49, 119, \cdots$

Obviously, this method can be applied iteratively and the successive sequences of partial sums would be obtained:

$$\frac{x}{1-2x-x^2} \mapsto 0, 1, 2, 5, 12, 29, 70, \cdots$$

$$\frac{x}{1-x \ \ 1-2x-x^2} \mapsto 0, 1, 3, 8, 20, 49, 119, \cdots$$

$$\frac{x}{1-x \ ^2 \ 1-2x-x^2} \mapsto 0, 1, 4, 12, 32, 81, 200, \cdots$$

$$\frac{x}{1-x \ ^3 \ 1-2x-x^2} \mapsto 0, 1, 5, 17, 49, 130, 330, \cdots$$

$$\frac{x}{1-x \ ^4 \ 1-2x-x^2} \mapsto 0, 1, 6, 23, 72, 202, 532, \cdots$$

III. Convolution and Generating Function

Convolution of the sequences a_n and b_n is defined as the sequence $\left\{\sum_{j=0}^{n} a_j b_{n-j}\right\}$. In fact, the convolution coincides with the sequence of coefficients of the product of two polynomials. Convolution of the sequences A and B is indicated as A*B.

Example 4

If $A = 0,1,2,3,4,5,\ldots$ and $B = 1,2,4,8,16,32,\ldots$, the first few elements of A*B are

$$0\cdot 1 = 0, \quad 0\cdot 2 + 1\cdot 1 = 1, \quad 0\cdot 4 + 1\cdot 2 + 2\cdot 1 = 4, \quad 0\cdot 8 + 1\cdot 4 + 2\cdot 2 + 3\cdot 1 = 11$$

$$0\cdot 16 + 1\cdot 8 + 2\cdot 4 + 3\cdot 2 + 4\cdot 1 = 26, \quad 0\cdot 32 + 1\cdot 16 + 2\cdot 8 + 3\cdot 4 + 4\cdot 2 + 5\cdot 1 = 57, \ldots.$$

That is $A*B = 0,1,4,11,26,57,\ldots$

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Convolution verifies commutative, associative, and distributive properties: $A*B=B*A$, $A*(B*C)=(A*B)*C$, and $A*(B+C)=A*B+A*C$.

3.1. *Theorem 2*

If f(x) and g(x) are the respective generating functions of the sequences $A= a_n$ and $B= b_n$, then $f(x)\cdot g(x)$ is the generating function of the convolution A*B.

Proof.

Proof is easy because $A(x)=\sum_{j\geq 0} a_j x^j$ and $B(x)=\sum_{j\geq 0} b_j x^j$ and so $A(x)\cdot B(x)=\sum_{j\geq 0}\left(\sum_{i=0}^{j} a_i b_{j-i}\right)x^j$ and the coefficients of A(x) · B(x) are the elements of the convolution sequence A*B.

Example 5

From Formula (4), $f(x)=\frac{1}{1-x}$ is the generating function of the constant sequence {1} = {1, 1, 1, …}. Then

$$\frac{1}{1-x}\mapsto 1^{(1)} = 1,1,1,1,...$$

$$\frac{1}{(1-x)^2}\mapsto 1^{(2)}=1^{(1)}*1^{(1)} = 1,2,3,4,5,... =\binom{n}{1}\text{: Linear numbers.}$$

$$\frac{1}{(1-x)^3}\mapsto 1^{(3)}=1^{(1)}*1^{(2)} = 1,3,6,10,15,... =\binom{n+1}{2}\text{: Triangular numbers.}$$

$$\frac{1}{(1-x)^4}\mapsto 1^{(4)}=1^{(1)}*1^{(3)} = 1,4,10,20,35,... =\binom{n+2}{3}\text{: Tetrahedral numbers.}$$

Example 6

Let us consider the generating functions $f(x)=\frac{1}{(1-x)^2}\mapsto 1,2,3,4,5,...$ and $g(x)=\frac{1}{1-kx-x^2}\mapsto F_{k,1},F_{k,2},F_{k,3},...$.

Then $f(x)\cdot g(x)=\frac{1}{(1-x)^2(1-kx-x^2)}\mapsto\left\{b_n=\sum_{j=1}^{n} j\,F_{k,j+1}\right\}$

$$= 1F_{k,1}, 1F_{k,2}+2F_{k,1}, 1F_{k,3}+2F_{k,2}+3F_{k,1}, 1F_{k,4}+2F_{k,3}+3F_{k,2}+4F_{k,1},...$$

$$= 1, k+2, k^2+2k+4, k^3+2k^2+5k+6, k^4+2k^3+6k^2+8k+9,...$$

For the classical Fibonacci sequence (k = 1) $g(x)=\frac{1}{1-x-x^2}$ so $f(x)\cdot g(x)\mapsto 1,3,7,14,26,...$ and $f(x)^2\cdot g(x)\mapsto 1,5,16,41,92,...$

Example 7

Let us consider the generating functions $f(x)=\frac{x}{1-x-x^2}\mapsto F_{n\ n\in N} = 0, 1, 1, 2, 3, 5, 8, 13,...$ and

$g(x)=\frac{1}{1-2x} \mapsto \ 2^n \ _{n\in N} = \ 1,2,4,8,16,32,\dots$. Then $f(x)\cdot g(x)=\frac{x}{1-x-x^2 \quad 1-2x} \mapsto \left\{\sum_{j=0}^{n} 2^{n-j}F_j\right\}_{n\geq 0}$

$= \ 0, F_1, 2F_1+F_2, 4F_1+2F_2+F_3, 8F_1+4F_2+2F_3+F_4,\dots \ = \ 0,1,3,8,19,43,94,\dots$

Example 8

From Example 5, $f(x)=\frac{x}{1-x^{\ 2}}$ is the generating function of the sequence of natural numbers $N=$ $0,1,2,3,4,5,\dots$ and from Formula (4) $g(x)=\frac{1}{1-2x}$ is that of $1,2,4,8,16,32,\dots$. Then $f(x)\cdot g(x)=\frac{x}{(1-x)^2(1-2x)}$ is the generating function of the convolution sequence $A*B= \ 0,1,4,11,26,57,..$ (Example 4).

3.2. *Self Convolution*

Convolution of a sequence with itself is called self convolution. Obviously, if $f(x)$ is the generating function of the sequence $\{a_n\}$, then $(f(x))^2$ is the generating function of the self convolution of $\{a_n\}$ that is

$$a_n \ ^2 = \ a_0^2, 2a_0a_1, 2a_0a_2+a_1^2, 2a_0a_3+2a_1a_2, 2a_0a_4+2a_1a_3+a_2^2,\dots \qquad (7)$$

Example 9

Find the sequence $\{a_n\}$ if {1, 4, 12, 30, 66, 132, 245, …} is its self convolution.

From Formula (7),

$a_0^2=1\rightarrow a_0=1$

$2a_0a_1=4\rightarrow a_1=2$

$2a_0a_2+a_1^2=12\rightarrow 2a_2+4=12\rightarrow a_2=4$

$2a_0a_3+2a_1a_2=30\rightarrow 2a_3+16=30\rightarrow a_3=7$

$2a_0a_4+2a_1a_3+a_2^2=66\rightarrow 2a_4+28+16=66\rightarrow a_4=11$

…….

$a_n \ = \ 1,2,4,7,11,16,22,29,\dots$

Example 10

Find the generating function of the previous sequence.

$1,2,4,7,11,16,22,29,\dots \ = \ 1,2,3,4,5,6,\dots \ + \ 0,0,1,3,6,10,\dots$

From Example 3, the generating function of the sequence $1,2,3,4,5,6,\dots$ is $f(x)=\frac{1}{(1-x)^2}$ and that of $0,0,1,3,6,10,\dots$ is $g(x)=\frac{x^2}{(1-x)^3}$. So, the generating function of the initial sequence is $f(x)+g(x)=\frac{1-x+x^2}{(1-x)^3}$

IV. GENERATING FUNCTIONS AND DERIVATIVES

By using the derivatives of the generating function of a sequence, we can find the generating function of some sequences related with the first one [5].

4.1. *Theorem 3*

If f(x) is the generating function of the sequence $a_n \ _{n\in N}$, then $f^{(r}(x)$ is the generating function of the sequence $\left\{ r!\binom{n}{r} a_n \right\}_{n\geq r}$ where $f^{(r}(x)$ is the rth derivative of f(x). So, $f(x) \mapsto a_n \ _{n\geq 0} \Rightarrow \left(f^{(r}(x) \mapsto \left\{ r!\binom{n}{r} a_n \right\}_{n\geq r} \right)$

Proof.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{j=0} a_j x^j$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = \sum_{j=1} j\, a_j x^{j-1}$$

$$f''(x) = a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots = \sum_{j=2} j(j-1)\, a_j x^{j-2}$$

$f'''(x) = a_3 + 4\cdot 3\cdot 2a_4x + 5\cdot 4\cdot 3a_5x^2 + 6\cdot 5\cdot 4a_6x^3 + \cdots = \sum_{j=3} j(j-1)(j-2)a_j x^{j-3}$ ….from where the theorem is proven.

Example 11

$f(x) = \dfrac{x}{1-2x-x^2}$ is the generating function of the Pell numbers {0, 1, 2, 5, 12, 29, 70, ..} (Formula (6)). Then

$$f(x) = \frac{x}{1-2x-x^2} \mapsto P_n \ _{n\geq 0} = 0, 1, 2, 5, 12, 29, 70, ...$$

$$f'(x) = \frac{1+x^2}{\left(1-2x-x^2\right)^2} \mapsto n\, P_n \ _{n\geq 1} = 1, 4, 15, 48, 145, 420, ...$$

$$f''(x) = \frac{2(2+3x+x^3)}{\left(1-2x-x^2\right)^3} \mapsto n(n-1)\, P_n \ _{n\geq 2} = 4, 30, 144, 580, 2100, 7098, ...$$

$$f'''(x) = \frac{6(5+8x+6x^2+x^4)}{\left(1-2x-x^2\right)^4} \mapsto n(n-1)(n-2)\, P_n \ _{n\geq 3} = 30, 288, 1740, 8400, ...$$

4.2. *Theorem 4*

If $f(x) \mapsto a_n$, then $D(x\, f(x)) \mapsto (n+1)\, a_n \ _{n\geq 0}$, being $D = \dfrac{d}{dx}$

Proof.

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

$$x\,f(x)=a_0x+a_1x^2+a_2x^3+a_3x^4+\cdots$$

$$D(x\,f(x))=a_0+2a_1x+3a_2x^2+4a_3x^3+\cdots\to\ D(x\,f(x))\mapsto\ (n+1)\,a_n\ _{n\geq 0}$$

This theorem can be applied iteratively:

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$
$$\to f_1(x)=D(x\,f(x))=a_0+2a_1x+3a_2x^2+4a_3x^3+\cdots\to\ D(x\,f(x))\mapsto\ (n+1)\,a_n\ _{n\geq 0}$$

$$x\,f_1(x)=a_0x+2a_1x^2+3a_2x^3+4a_3x^4+\cdots$$
$$\to f_2(x)=D(x\,f_1(x))=a_0+2^2a_1x+3^2a_2x^2+4^2a_3x^3+\cdots\to\ D(x\,f_1(x))\mapsto\ (n+1)^2\,a_n\ _{n\geq 0}$$

$$x\,f_2(x)=a_0x+2^2a_1x^2+3^2a_2x^3+4^2a_3x^4+\cdots$$
$$\to f_3(x)=D(x\,f_2(x))=a_0+2^3a_1x+3^3a_2x^2+4^3a_3x^3+\cdots\to\ D(x\,f_2(x))\mapsto\ (n+1)^3\,a_n\ _{n\geq 0}$$

Example 12

From Equation (4), $f(x)=\dfrac{1}{1-x}\mapsto\ 1\ _{n\geq 1}$. Then

$$f_0(x)=f(x)=\frac{1}{1-x}\mapsto\ 1\ _{n\geq 1}$$

$$f_1(x)=D(x\,f_0(x))=\frac{1}{(1-x)^2}\mapsto\ n+1\ _{n\geq 0}\ =\ n\ _{n\geq 1}$$

$$f_2(x)=D(x\,f_1(x))=\frac{1+x}{(1-x)^3}\mapsto\ (n+1)^2\ _{n\geq 0}\ =\ n^2\ _{n\geq 2}$$

$$f_3(x)=D(x\,f_2(x))=\frac{1+4x+x^2}{(1-x)^4}\mapsto\ n^3\ _{n\geq 3}$$

$$f_4(x)=D(x\,f_3(x))=\frac{1+11x+11x^2+x^3}{(1-x)^5}\mapsto\ n^4\ _{n\geq 4}$$

With help of Mathematical® it is easy to find out the preceding numerical sequences:

$$f[0,x\_]:=\frac{1}{1-x}$$

$$f[r\_,x\_]:=D[x\,f[r-1,x],x]$$

Table [Coefficient List [Series [f [r, x], [x, 0, 25], x], {r, 0, 5}]

V. Relation between the Generating Function and the Recurrence Relation

There is an intimate relationship between the recurrence relation in a sequence and its generating function.

5.1. *Theorem 5*

Let us consider the numerical sequence $\{U_n\}_{n\in N}$ with the recurrence relation $U_n = a_1U_{n-1} + a_2U_{n-2} + \cdots + a_rU_{n-r}$ and the initial conditions $U_0, U_1, U_2, \ldots U_{r-1}$.

The generating function of the sequence $\{U_n\}$ is $f(x)=\dfrac{N(x)}{P(x)}$ with $N(x)=\sum_{j=0}^{r-1}\left(U_j-\sum_{i=1}^{j}a_iU_{j-i}\right)x^j$ and $P(x)=1-\sum_{j=1}^{r-1}a_jx^j$

Proof.

$$f(x)=U_0+U_1x+U_2x^2+U_3x^3+\cdots+U_{n-1}x^{n-1}+U_nx^n+\cdots$$

$$a_1x\,f(x)=a_1U_0x+a_1U_1x^2+a_1U_2x^3+\cdots+a_1U_{n-2}x^{n-1}+a_1U_{n-1}x^n+\cdots$$

$$a_2x^2f(x)=a_2U_0x^2+a_2U_1x^3+a_2U_2x^4+\cdots+a_2U_{n-3}x^{n-1}+a_2U_{n-1}x^n+\cdots$$

……………….

$$a_{r-1}x^{r-1}f(x)=a_{r-1}U_0x^{r-1}+a_{r-1}U_1x^r+a_{r-1}U_2x^{r+1}+\cdots+a_{r-1}U_{n-r}x^{n-1}+a_{r-1}U_{n-r+1}x^n+\cdots$$

$$a_rx^rf(x)=a_rU_0x^r+a_rU_1x^{r+1}+a_rU_2x^{r+2}+\cdots+a_rU_{n-r-1}x^{n-1}+a_rU_{n-r}x^n+\cdots$$

$$\left(1-\left(a_1x+a_2x^2+\cdots+a_{r-1}x^{r-1}\right)\right)f(x)=U_0+\left(U_1-a_0U_0\right)x+\left(U_2-\left(a_1U_1+a_2U_0\right)\right)x^2+$$
$$+\left(U_3-\left(a_1U_2+a_2U_1+a_3U_0\right)\right)x^3+\cdots+\left(U_{n-1}-\left(a_1U_{n-2}+a_2U_{n-3}+\cdots+a_{r-1}U_{n-r}\right)\right)x^{n-1}$$

because the rest of the addends cancel due the recurrence relation. So the theorem.

Example 13

Let us consider the sequence of squares of the k-Fibonacci numbers $\{F_{k,n}^2\}$. This sequence verifies the recurrence relation $F_{k,n}^2=(k^2+1)F_{k,n-1}^2+(k^2+1)F_{k,n-2}^2-F_{k,n-3}^2$ Formula (1), with the initial conditions $F_{k,1}^2=1,\ F_{k,2}^2=k^2,\ F_{k,3}^2=(k^2+1)^2$. We will apply this last theorem to find out the generating function of the sequence $\{F_{k,n}^2\}$:

$$N(x)=U_0+\left(U_1-a_1U_0\right)x+\left(U_2-\left(a_1U_1+a_2U_0\right)\right)x^2$$

$$U_0=F_{k,1}^2=1,\ U_1=F_{k,2}^2=k^2,\ U_2=F_{k,3}^2=(k^2+1)^2$$

$$N(x)=1+\left(k^2-\left(k^2+1\right)\right)x+\left(\left(k^2+1\right)^2-\left(\left(k^2+1\right)k^2+\left(k^2+1\right)1\right)\right)x^2=1-x$$

$$P(x)=1-a_1x-a_2x^2+a_3x^3=1-\left(k^2+1\right)x-\left(k^2+1\right)x^2+x^3$$

$$f(x)=\frac{1-x}{1-\left(k^2+1\right)\left(x+x^2\right)+x^3}$$ what is Equation (3).

Remark

Given the recurrence relation $U_n = a_1U_{n-1} + a_2U_{n-2} + \cdots + a_rU_{n-r}$ of the sequence $\{U_n\}$, its characteristic polynomial is $1 - a_1 x - a_2 x^2 - \cdots - a_r x^r$ that is the denominator of the generating function of $\{U_n\}$. And vice versa.

Example 14

Let $f(x) = \dfrac{1-2x+3x^2}{1-3x+4x^2-2x^3}$ be the generating function of the sequence $\{U_n\}$. Find the recurrence relation between its elements.

From the denominator $1 - 3x + 4x^2 - 2x^3$, equation $x^0 - 3x + 4x^2 - 2x^3 = 0$ is obtained. If $x^r = U_{n-r}$ then $U_n - 3U_{n-1} + 4U_{n-2} - 2U_{n-3} = 0$, from where the recurrence relation $U_n = 3U_{n-1} - 4U_{n-2} + 2U_{n-3}$. Because the denominator is a polynomial of degree 3, we must find three initial conditions:

$$\begin{array}{l|l} 1-2x+3x^2 & 1-3x+4x^2-2x^3 \\ \hline \underline{-1+3x-4x^2+2x^3} & 1+x+2x^2+\cdots \\ \quad x-x^2+2x^3 & \\ \quad \underline{-x+3x^2-4x^3+2x^4} & \\ \qquad 2x^2-2x^3+2x^4 & \end{array}$$

The three first terms of the generated sequence are 1, 1, 2, so the initial conditions for this sequence are $U_0 = 1$, $U_1 = 1$, and $U_2 = 2$.

Using either the generating function or the recurrence relation, the first terms of the numerical sequence are {1, 1, 2, 4, 6, 6, 2, – 6, –14, –14, 2, 34, 66, …}

VI. On the Convolutive Inverse Sequence

The convolutive identity is the sequence I = {1, 0, 0, 0, …} and verifies I * A = A * I = A.

A^{-1} is the convolutive inverse sequence of A if $A * A^{-1} = A^{-1} * A = I$. Logically $\left(A^{-1}\right)^{-1} = A$

It is evident that every sequence of the form U = $\{u_0, u_1, u_2, u_3, \ldots\}$ admits inverse if $U_0 \neq 0$. So the convolutive inverse sequence is integer, we will take $U_0 = 1$. Let $U^{-1} = \{b_0, b_1, b_2, \ldots\}$ be the convolutive inverse sequence of U, that is $U * U^{-1} = I$. Then must be

$$b_0 = 1,$$

$$b_1 + u_1 b_0 = 0 \to b_1 = -u_1,$$

$$b_2 + u_1 b_1 + u_2 b_0 = 0 \to b_2 = u_1^2 - u_2,$$

$$b_3 = -u_1^3 + 2u_1u_2 - u_3,$$

$$b_4 = u_1^4 - 3u_1^2u_2 + 2u_1u_3 + u_2^2 - u_4, \ \ldots$$

Example 15

Find the convolutive inverse sequence of the sequence $\{u_n\}$ such that $u_0 = 1$, $u_1 = 3$, $u_2 = 6$ and $u_n = u_{n-1} + 2u_{n-2} - u_{n-3}$

Easily we can find that $\{u_n\}$ = {1, 3, 6, 11, 20, 36, 65, 117, 211, 380, 685, 1234…}. Then

$b_0 = 1$,

$3 + b_1 = 0 \to b_1 = -3$,

$6 + 3(-3) + b_2 = 0 \to b_2 = 3$,

$11 + 6(-3) + 3 \cdot 3 + b_3 = 0 \to b_3 = -2$,

$20 - 33 + 18 - 6 + b_4 = 0 \to b_4 = 1$ …

Then U^{-1} = {1, – 3, 3, – 2, 1, 0, 1, 4, 21 …}

So, the set of numerical sequences $u_n \;_{n\in N}$ with $u_0 \neq 0$, with respect to the sum and the convolution is an abelian field.

6.1. *Finite Inverse Convolutive Sequence*

Given the recurrence relation $u_n = \sum_{i=0}^{r-1} a_i u_{n-i}$, to find the sequence it generates it is necessary to indicate *r* initial conditions, that is, the values of the first *r* terms of the sequence. These values can be arbitrary, but if they are chosen in such a way that, if $u_n = 0$, $\forall n<0$, they also verify this relation, then it is verified that the elements b_n of convolutive inverse sequence are null for $n \geq r$. Moreover, the convolutive inverse sequence has the form $U^{-1} = \{1, a_1, a_2, a_3, \ldots a_{r-1}, 0, 0, 0, \ldots\}$

Example 16

Find the convolutive inverse sequence of {1, 2, 5, 5, 1, -7, 15, -15, 1, 33, 65, …}

Recurrence relation of this sequence is $u_n - 3u_{n-1} + 4u_{n-2} - 2u_{n-3} = 0$ and u_0, u_1, u_2, and u_3 verify also this relation, so U^{-1} = {1, – 3, 4, – 2, 0, 0, 0 …}

Example 17

Find the convolutive inverse sequence of {1, 2, – 3, – 5, 1, 0, 0, 0, …}

Recurrence relation of the requested sequence is $u_n = 2 - 2u_{n-1} + 3u_{n-2} + 5u_{n-3} - u_{n-4}$ and the first five elements also verifies this relation, so U = {1, – 2, 7, – 15, 40, – 88, 214, …}

The abelian field of invertible sequences is disjoint union of the set of sequences with infinite elements whose first *r* terms do no verify the recurrence relation and the set of sequences with a maximum of *r* non-zero elements.

6.2. *Inverse of a Convolutiion*

Inverse of a convolution is the convolution of the inverses: $(U * V)^{-1} = U^{-1} * V^{-1}$.

Proof.

$(U * V) * (U * V)^{-1} = I$

$(U * V) * (V^{-1} * U^{-1}) = U * (V * V^{-1}) * U^{-1} = U * I * U^{-1} = U * U^{-1} = I \to (U * V)^{-1} = V^{-1} * U^{-1} = U^{-1} * V^{-1}$

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Author's Profile

Sergio Falcon, Department of Mathematics, University of Las Palmas de Gran Canaria, 35004 Las Palmas, Spain.