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Horizontal Flight of a Rocket in Homogenous Atmosphere

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***Abstract* – In this paper flight performance of a rocket in horizontal plane in homogeneous atmosphere is studied taking into account the propellant consumption leading to decreasing mass of the rocket in flight .So much so that it encounters horizontal flight first with constant thrust, second with constant thrust acceleration and third with thrust obeying a power law. Horizontal flight with time-varying velocities under controlled thrust is also incorporated.**

***Keywords* – Horizontal Flight, Rocket, Homogenous Atmosphere.**

I. Introduction

Angelo Miele[1] discussed horizontal flight of an aircraft / rocket in form of level flight and curvilinear flight in horizontal plane but excluding mass variation caused by propellant consumption. He[1] innovated a problem of accelerating an aircraft from one velocity to another velocity at a constant altitude, ie, in horizontal plane. Leitman[2] derived maximum range for a rocket in horizontal flight and studied an optimum burning program for horizontal flight. N.X. Vinit[4] innovated minimum fuel rocket flight in horizontal plane. J. Amer[5] considered optimum burning problem for horizontal flight.

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In this paper is investigated horizontal flights in homogeneous atmosphere respectively with constant thrust, constant thrust acceleration, thrust obeying a power law of the instantaneous mass and also horizontal flights respectively with constant tangential acceleration, with also exponential variation of the velocity and variable thrust program to overcome the parabolic drag. Mass variation in case of rocket/spacecraft due to propellant consumption is taken into consideration.

II. Flight of the Spacecraft in Horizontal Plane with Constant Thrust

Equations of motion of the spacecraft in atmosphere with drag proportional to the square of velocity in horizontal plane with as usual notations are written as $\frac{dx}{dt} = V\cos\theta$, $\frac{dy}{dt} = V\sin\theta$, $\frac{dV}{dt} = \frac{T-D}{m}$.

$$D = \frac{1}{2}C_D\rho SV^2 = K_D V^2,\ L = \frac{1}{2}C_L\rho SV^2,\ L\cos\mu = mg \tag{1}$$

$$mV\frac{d\theta}{dt} = L\sin\mu\ ,\ V\frac{d\theta}{dt} = g\tan\mu,\ T = -V_E\frac{dm}{dt} = V_E\beta \tag{2}$$

where (x, y) is the position of the rocket with regard to right-handed frame of axes XOY (OX at right angles to OY with O as the origin) lying on the horizontal plane, θ the inclination of velocity V along the tanget to the flight path with the x axis, at time t, D the drag, L the lift, T the constant thrust, m the mass, V_E the exhaust velocity, C_L lift coefficient, C_D the drag coefficient, S the reference area of the rocket, ρ the air density, g the acceleration due to gravity, μ the inclination of the lift with vertical, β the rate of propellant consumption and K_D

the drag factor.

Combining third one of set (1) with the second of (2) and simplifying we get

$$\frac{m}{V_E}\frac{dV}{dm} = -\frac{T-D}{T} \quad \text{(By use of (1))}$$

$D = K_D V^2$

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$$\frac{dm}{m} = \frac{c_0^2}{V_E}\frac{dV}{V^2-c_0^2},\ c_0^2 = \frac{T}{K_D} \tag{3}$$

which can be integrated using the initial conditions

At t = 0, m = m_0, V = V_0, x = 0, y = 0, θ = 0 (4)

$$\text{Log}\frac{m}{m_0} = \frac{c_0}{2V_E}\log\left(\frac{V-c_0}{V+c_0}\right)\left(\frac{V_0+c_0}{V_0-c_0}\right)$$

$$\text{Or, } m = m_0 c\left(\frac{V-c_0}{V+c_0}\right)^{\frac{1}{n}} \tag{5}$$

$$\text{Where } c^n = \frac{V_0+c_0}{V_0-c_0} > 1 \text{ and } \frac{1}{n} = \frac{c_0}{2V_E} \tag{6}$$

$$\text{Or, } V = c_0\frac{(cm_0)^n+(m)^n}{(cm_0)^n-(m)^n}\ (cm_0)^n > (m)^n \tag{7}$$

which indicates the velocity as a function of mass. Further, from (2) the time-varying mass is given by

$$m = m_0 - \beta t \tag{8}$$

which is substituted into (7) to yield the time- history of velocity: $V = c_0\frac{(cm_0)^n+(m_0-\beta t)^n}{(cm_0)^n-(m_0-\beta t)^n}$.

To maintain the horizontal flight with constant thrust ie constant propellant mass flow β, 5 th relationship of (1) is utilized by varying the angle of attack ie lift cefficient.

$$C_L(t) = \frac{2(m_0-\beta t)}{\rho S c_0^2 \cos\mu}\left(\frac{(cm_0)^n-(m_0-\beta t)^n}{(cm_0)^n+(m_0-\beta t)^n}\right)^2 g \tag{9}$$

Combining 6th of set(1) with (2)and (6), one gets,

$$\frac{d\theta}{dm} = \left(\frac{-gV_E\tan\mu}{Tc_0}\right)\frac{(cm_0)^n-m^n}{(cm_0)^n+m^n} = \left(\frac{-gV_E\tan\mu}{Tc_0}\right)\left(\frac{2(cm_0)^n}{(cm_0)^n+m^n}-1\right) \tag{10}$$

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The propellant mass consumption in case of a rocket or fuel consumption in case of an aircraft, being small compared to the gross weight of the rocket in flight, $\frac{m^n}{(cm_0)^n} < 1$ entailing Binomial expansion of (10) followed by term-by-term integration subject to the initial conditions (4) so that,

$$\text{Or,} \frac{d\theta}{dm} = -\left(\frac{gV_E\tan\mu}{Tc_0}\right)\left(\frac{2(cm_0)^n}{(cm_0)^n+m^n}-1\right) = \left(\frac{gV_E\tan\mu}{Tc_0}\right)\left(1-\frac{2}{1+\frac{m^n}{(cm_0)^n}}\right)$$

$$\text{Or, } \theta = \left(\frac{gV_E\tan\mu}{Tc_0}\right)\left[2\int_m^{m_0}\left\{1+\sum_1^\infty(-1)^r\left(\frac{m}{cm_0}\right)^{nr}\right\}dm-(m_0-m)\right]$$

Or, $\theta = \left(\frac{gV_E \tan\mu}{Tc_0}\right)\{(m_0 - m) + 2\sum_1^\infty (-1)^r (\frac{1}{cm_0})^{nr} \left(\frac{m_0^{nr+1}}{nr+1} - \frac{m^{nr+1}}{nr+1}\right)\}$ (11)

For integral values of n=1,2,3,4 the above equation gives close-form solution. However let us take up the case when n=4 in (10).

$$\theta_4 = \frac{gV_E}{Tc_o}(\tan\mu)\left\{\int_m^{m_0} \frac{2dm}{1+\frac{m^4}{(cm_0)^4}} - (m_0 - m)\right\} \quad (12)$$

Let us put $\frac{m}{cm_0} = x$ in the above integrand to evaluate $\int \frac{dx}{1+x^4}$,

To begin with, we separate $\frac{1}{1+x^4}$ into partial fractions with subtlety and intuition to facilitate integration in close form.

$$\frac{1}{1+x^4} = \frac{1}{2}\frac{2\frac{1}{x^2}}{x^2+\frac{1}{x^2}} = \frac{1}{2}\{\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2} - \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2}\} \text{ (Integrating with respect to x)}$$

$$\int \frac{dx}{1+x^4} = \int \frac{1}{2}\{\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2} - \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2}\}dx = \frac{1}{2}\{\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\log\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\} \quad (13)$$

Recalling the above substitution in (13) and then using the same in (12), we get,

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$$\theta_4 = \frac{gV_E}{Tc_o}(\tan\mu)[\{\frac{cm_0}{2\sqrt{2}}\log\{\left(\frac{(1+\sqrt{2}c+c^2)m_0^2}{(1-\sqrt{2}c+c^2)m_0^2}\right)(\frac{m^2-\sqrt{2}cmm_0+c^2m_0^2}{m^2+\sqrt{2}cmm_0+c^2m_0^2})\} - (m_0 - m)\}$$

$$+\frac{1}{\sqrt{2}}\{\tan^{-1}\frac{1}{\sqrt{2}}\left(\frac{m}{cm_0}-\frac{cm_0}{m}\right) - \tan^{-1}\frac{1}{\sqrt{2}}\left(\frac{1}{c}-c\right)\}] \quad (14)$$

Combining (7), (13) and (14) give the time-history of the heading and using (1), (2) and (9) we can find the longitudinal x and latitudinal y distances in time-integral form which can be evaluated numerically rather than in close form. By use of (3), (6) and third of (1) the total ie curvilinear distance traveled in the horizontal plane in form of an integral with respect to v has to be determined per se numerically such that.

$$\frac{dV}{dt} = V\frac{dV}{ds} \quad (15)$$

The foregoing analysis pertains to the situation where the thrust is less than the drag resulting in velocity gradually decreasing. Now let us take up a general vital case wherein the thrust plays dominating role over the drag and so equation (3) is rewritten as.

$$\frac{dm}{dV} = -\frac{C_0^2}{V_E}\frac{m}{C_0^2-V^2} \quad (16)$$

Or, $\int_m^{m_0}\frac{dm}{m} = \frac{C_0^2}{V_E}\int_{V_0}^{V}\frac{dV}{C_0^2-V^2}$

which gives the same type of relationship as earlier, ie, mass-velocity distribution.

$$m = m_0 c(\frac{c_0-V}{c_0+V})^{\frac{1}{n}} \quad (17)$$

where c is redefined as earlier.

III. Flight in Horizonatal Plane with Constant Thrust Acceleration

Unlike A.Miele[1] and other workers viz-a-viz contributors in Space Dynamics on account of propellant consumption, mass variation of the spacecraft is taken into consideration for horizontal flight and as such the equations of motion with.

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Constant thrust acceleration τ are written as (Using herein v in place of V arising in the earlier section).

$$\frac{dx}{dt} = v\cos\theta, \frac{dy}{dt} = v\sin\theta, \frac{dv}{dt} = \tau - \frac{K_D v^2}{m}, \frac{d\theta}{dt} = \frac{g}{v}\tan\mu \tag{18}$$

$$L = \frac{1}{2}C_L \rho S v^2 = K_L v^2, L\sin\mu = mv\frac{d\theta}{dt}, L\cos\mu = mg, -V_E\frac{dm}{dt} = m\,\tau \tag{19}$$

Having the initial conditions:

$$\text{At } t = 0, x = 0, y = 0, \theta = 0, m = m_0, \tag{20}$$

The last equation of set (19) gives,

$$m = m_0 e^{-\alpha t}, \alpha = \frac{\tau}{V_E} \tag{21}$$

Combining third of (18) with last of (19), we have,

$$\frac{dv}{dm} = -\frac{V_E}{m}\left(1 - \frac{K_D v^2}{m\tau}\right) \tag{22}$$

which is rewritten in form of Riccati's equation[3]

$$\frac{dv}{dm} = P + Qv + Rv^2 = P + Rv^2 \tag{23}$$

where P and Q are functions of the independent variable m: $P == -\frac{V_E}{m},\ Q = 0,\ R = \frac{V_E K_D}{m^2 \tau}$

Equation (23) can be converted into Bessel's[3] equation with

$$v = -\frac{1}{Ru}\frac{du}{dm} = -\frac{-\tau m^2}{K_D V_E}\frac{1}{u}\frac{du}{dm} \tag{24}$$

$$m^2\frac{d^2u}{dm^2} + 2m\frac{du}{dm} - \frac{K_D V_E^2}{m\tau} = 0, \tag{25}$$

when compared with[3]

$$x^2\frac{d^2y}{dx^2} + nx\frac{dy}{dx} + (b + cx^{2m'})y = 0$$

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Actually m in the textbook[3] is replaced by m′. We find,

$$x = m, y = u, n = 2, b = 0, c = -\frac{K_D V_E^2}{\tau}, m' = -\frac{1}{2} \tag{26}$$

and with relationship[3],

$$\mu^2 m'^2 = \frac{1}{4}(n-1)^2 - b, \quad \text{one gets } \mu = \pm 1 \tag{27}$$

Hence solution to (25) is obtained in the textbook[3] as,

$$y = x^{\frac{-1}{2}}(n-1)\{AJ_{\mu}\left(\frac{c^{\frac{1}{2}}x^{m'}}{m'}\right) + Bj_{-\mu}\left(\frac{c^{\frac{1}{2}}x^{m'}}{m'}\right)\} \quad (28)$$

For integral values = $\mu = \pm 1,\ j_{-1}(x) = Y_1(x)$ so that,

$$u = m^{\frac{-1}{2}}\left\{AJ_1\left(-2V_E\sqrt{-\frac{K_D}{m\tau}}\right) + Bj_{-1}\left(-2V_E\sqrt{-\frac{K_D}{m\tau}}\right)\right\} \quad (29)$$

which is in complex form, but can be put in real form:

$$u = m^{\frac{-1}{2}}\left\{AI_1\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right) + BY_1\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)\right\} \quad (30)$$

which, when substituted in (24) yields

$$v(m) = \frac{m\tau}{2K_D V_E} - \sqrt{\frac{m\tau}{K_D}}\frac{I_1'\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)+\lambda Y_1'\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)}{I_1\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)+\lambda Y_1\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)} \quad (31)$$

where $\lambda = \frac{B}{A}$ is evaluated with the initial conditions (20) and the initial velocity v_0 *is* determined by replacing m by m_0 in (31). Using (31) in (18)is obtained:

$$\theta = \frac{gV_E \tan\mu}{\tau}\int_m^{m_0}\left[\frac{m\tau}{2K_D V_E} - \sqrt{\frac{m\tau}{K_D}}\frac{I_1'\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)+\lambda Y_1'\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)}{I_1\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)+\lambda Y_1\left(-2V_E\sqrt{\frac{K_D}{m\tau}}\right)}\right]^{-1} dm \quad (32)$$

The position of the rocket at time t in the horizontal plane is obtained as,

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$$x = \int_m^{m_0}\frac{V_E v(m)\cos\theta(m)dm}{\tau m},\ y = \int_m^{m_0}\frac{V_E v(m)\sin\theta(m)dm}{\tau m} \quad (33)$$

which need to be evaluated numerically with given data in cumbersome process. For such horizontal flight, the vertical component of the lift has to be balanced by the weight of the rocket and as such the angle of attack ie lift coefficient turns out to be a control parameter with regard to time t due to (19) and (21):

$$\frac{1}{2}C_L(t)\rho S v^2(t)\cos\mu = m_0 g e^{-\alpha t} \quad (34)$$

IV. Horizontal Flight with Variable Thrust Obeying Power Law of Instantaneous Mass

In view of the above insight the thrust of the spacecraft with mass m at any instant of time is

$$T = T_0\left(\frac{m_0}{m}\right)^n = -V_E\frac{dm}{dt} \quad (35)$$

Or, equivalently $T = m\tau = m_0\tau_0\left(\frac{m_0}{m}\right)^n$

Instantaneous thrust acceleration = $\tau = \tau_0\left(\frac{m_0}{m}\right)^{n+1}$ (36)

where T_0 and τ_0 are respectively initial thrust and initial thrust acceleration. However, (35) and (36) corroborate that for constant thrust, n = 0 and for constant thrust acceleration n = −1. Solving equation (35) by using the initial conditions (20) is obtained the mass variation law in this case as.

$$m = m_0[1 - \frac{t(n+1)T_0}{V_E m_0}]^{\frac{1}{n+1}} \qquad (37)$$

Now in the light of the previous case the relevant equation of motion becomes,

$$\frac{dv}{dm} = -\frac{V_E}{m}(1 - \frac{K_D v^2}{T_0}\frac{m^n}{{m_0}^n}) \qquad (38)$$

On the same line as in the previous case with the substitution,

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$$v = -\frac{T_0 m_0^n}{V_E K_D m^{n-1} u}\frac{du}{dm} \qquad (39)$$

Equation (38) reduces to a form, ie, Bessel's equation,

$$m^2\frac{d^2u}{dm^2} - (n-1)m\frac{du}{dm} - \frac{K_D {V_E}^2}{T_0}\frac{m^n u}{{m_0}^n} = 0 \qquad (40)$$

Adopting the same technique as earlier, vide Forsyth[3] ,we have for equation (40) the following general solution,

$$u = m^{\frac{n}{2}}\left\{AJ_1\left(2\frac{V_E}{n}\sqrt{-\frac{K_D m^n}{T_0 m_0^n}}\right) + BY_1\left(2\frac{V_E}{n}\sqrt{-\frac{K_D m^n}{T_0 m_0^n}}\right)\right\} \qquad (41)$$

In consequence of (41) and (39) the velocity at time t is given by,

$$v(m) = \frac{T_0(\frac{m_0}{m})^n}{2K_D V_E} - \sqrt{\frac{T_0(\frac{m_0}{m})^n}{K_D}}\frac{I_1'\left(2\frac{V_E}{n}\sqrt{\frac{K_D m^n}{T_0 m_0^n}}\right)+\lambda K_1'\left(2\frac{V_E}{n}\sqrt{\frac{K_D m^n}{T_0 m_0^n}}\right)}{I_1\left(2\frac{V_E}{n}\sqrt{\frac{K_D m^n}{T_0 m_0^n}}\right)+\lambda K_1\left(2\frac{V_E}{n}\sqrt{\frac{K_D m^n}{T_0 m_0^n}}\right)} \qquad (42)$$

which can be expressed as a function of time t by use of (21). The rest of the analysis is in the same line as in the previous case. It can be noted that if n = −1, the present case leads to a constant thrust acceleration program. But if n tends to 0 (n = 0), it gives constant thrust program.

V. Flight with Constant Tangential Acceleration in Horizontal Plane

In this feature the preceding equation of motion changes as follows:

$$\frac{dv}{dt} = \frac{T-D}{m} = f = \text{tangential acceleration} \qquad (43)$$

Combining (43) with the thrust equation (35) is obtained,

$$\frac{dm}{dv} + \frac{m}{V_E} = \frac{-K_D v^2}{f V_E} \qquad (44)$$

Solution to (44) with the same initial conditions (20) is obtained as,

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$$me^u - m_0 e^{u_0} = -\frac{K_D {V_E}^2}{f}\int_{u_0}^{u} e^u u^2 du$$

Assuming $u = \frac{v}{V_E}$ and $u_0 = \frac{v_0}{V_E}$, (45)

$$m = m_0 e^{-(u-u_0)} - e^{-u}\frac{K_D V_E{}^2}{f}(u^2 - 2u - u_0{}^2 + 2u_0)$$

Or, $m = m_0 e^{-(u-u_0)} - \frac{K_D V_E{}^2}{f}(u - u_0)(u + u_0 - 2)e^{-u} = m(t)$, (46)

The following relationship is obtained from equation (43),

$v = v_0 + ft \;\; ie \; u = u_0 + f\frac{ft}{V_E}$, (47)

Further equation (43) when combined with the heading equation of set (1),

$$\frac{d\theta}{dv} = \frac{g\tan\mu}{vf} \tag{48}$$

With the same initial conditions as earlier,

$$\theta = \frac{g\tan\mu}{f}\log\frac{v}{v_0} \tag{49}$$

Or, $\frac{dx}{dv} = \frac{v\cos\theta}{f} = \frac{v\cos\left(\frac{g\tan\mu}{f}\log\frac{v}{v_0}\right)}{f}$, Letting $v = v_0 e^{\epsilon}$ and $c = \frac{g\tan\mu}{f}$ (50)

$$\frac{fx}{v_0{}^2} = \int_0^{\epsilon} e^{2\epsilon}\cos(c\,\epsilon)\,d\epsilon = I_1 \tag{51}$$

Similarly,

$$\frac{fy}{v_0{}^2} = \int_0^{\epsilon} e^{2\epsilon}\sin(c\,\epsilon)\,d\epsilon = I_2 \tag{52}$$

After evaluations of the above integrals by parts, the horizontal distances of the space craft are given by,

$$x = \frac{v_0^2}{f}\frac{e^{2\epsilon}\{2\cos(c\epsilon) + c\sin(c\epsilon)\} - 2}{c^2+4}\;;\; y = \frac{v_0^2}{f}\frac{e^{2\epsilon}\{2\sin(c\epsilon) - c\cos(c\epsilon)\} + c}{c^2+4} \tag{53}$$

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However the time-varying angle of attack ie lift coefficient is to be maintained as a control parameter, by use of relationship (18), from equations (46) and (47).

$$\frac{1}{2}C_L(t)\rho S(v_0 + ft)^2\cos\mu = g\{m_0 e^{-\frac{ft}{V_E}} - K_D V_E t\left(2u_0 + \frac{ft}{V_E} - 2\right)\} \tag{54}$$

The arc length distance (curved path) traveled by the spacecraft in time t is derived with the help of the initial conditions (20).

$$\frac{vdv}{ds} = f \tag{55}$$

Or, $s = \frac{v^2 - v_0^2}{2s}$, $s = v_0 t + \frac{1}{2}ft^2$ (56)

VI. Horizontal Flight with Exponential Variation of Velocity with Time in Atmosphere of Resistance Proportional to Square of the Velocity

In this section we look for a thrust program vis-à-vis rocket mass variation law leading to exponential variati-

-on of velocity with time. With as usual notation in the nomenclature we have,

$v = v_0 e^{\alpha t}$, $\alpha > 0$ for increasing velocity (57)

$\frac{dv}{dt} = \alpha v$, $\alpha < 0$ for decreasing velocity

Or, $u = u_0 e^{\alpha t}$, where $u = v/V_E$ and $u_0 = v_0 / V_E$ (58)

The equations of motion as a consequence of such maneuvering are,

$x = v\cos\theta$, $y = v\sin\theta$, $D = \frac{1}{2} C_D \rho S v^2$, $L\cos\mu = \frac{1}{2} C_L \rho S v^2 \cos\mu = mg$ (59)

$L\sin\mu = v\frac{d\theta}{dt}$, $-V_E \frac{dm}{dt} = T(m)$ or $T(t)$ (60)

By use of (18), (19), (20), one gets the vital equation, $m\frac{dv}{dt} = T(t) - K_D v^2$

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Or, $\alpha v m = -V_E \frac{dm}{dt} - K_D v^2$ (61)

Or, $\frac{dm}{dv} + \frac{m}{V_E} = -\frac{K_D v}{\alpha V_E}$ (62)

whose solution subject to the initial conditions(20) in the previous section and by use of (58) is given by,

$m = m_0 e^{-(u-u_0)} + \frac{K_D}{\alpha} V_E \{(1-u)e^{u} - (1-u_0)e^{u_0}\} e^{-u}$ (63)

Or, the mass as a function of time t is obtained due to (58) and (63):

$m = m_0 e^{u_0(1-e^{\alpha t})} + \frac{K_D}{\alpha} V_E \{(1 - u_0 e^{\alpha t}) e^{u_0 e^{\alpha t}} - (1-u_0)e^{u_0}\} e^{-u_0 e^{\alpha t}}$ (64)

We can also determine the other aspects as done earlier :

$\frac{d\theta}{dv} = \frac{g}{\alpha v^2} \tan\mu$ (65)

so that because of the same initial conditions we get

$\theta = \frac{g}{\alpha} \tan\mu \int_{v_0}^{v} \frac{dv}{v^2} = \frac{g}{\alpha} (\tan\mu)(\frac{1}{v_0} - \frac{1}{v})$ (66)

$\frac{dx}{dv} = \frac{\cos\theta}{\alpha} = \frac{\cos\{\frac{g}{\alpha}(\tan\mu)\left(\frac{1}{v_0} - \frac{1}{v}\right)\}}{\alpha} = \frac{\cos c_0 \cos\frac{\lambda}{v} + \sin c_0 \sin\frac{\lambda}{v}}{\alpha}$ (67)

where $c_0 = \frac{g\tan\mu}{\alpha v_0}$ and $\lambda = \frac{g\tan\mu}{\alpha}$ (68)

Solution to (67) subject to the initial conditions as mentioned earlier becomes,

$x = \frac{1}{\alpha} \int_{v_0}^{v} \cos(c_0 - \frac{\lambda}{v}) dv = \frac{I_1 \cos c_0 + I_2 \sin c_0}{\alpha}$ (69)

$y = \frac{1}{\alpha} \int_{v_0}^{v} \sin(c_0 - \frac{\lambda}{v}) dv = \frac{I_1 \sin c_0 - I_2 \cos c_0}{\alpha}$ (70)

where $I_1 = \int_{v_0}^{v} \cos\frac{\lambda}{v} dv$ *and* $I_2 = \int_{v_0}^{v} \sin\frac{\lambda}{v} dv$ (71)

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To determine integrals (71) let us substitute $\frac{\lambda}{v} = \omega$ *so that* $\frac{\lambda}{v^2} dv = -d\omega$.

Then $I_1 = -\lambda \int_{\omega_0}^{\omega} \frac{\cos\omega}{\omega^2} d\omega = \lambda\{(\frac{\cos\omega}{\omega} - \frac{\cos\omega_0}{\omega_0}) + \int_{\omega_0}^{\omega} \frac{\sin\omega}{\omega} d\omega$.

$$I_1 = \lambda\{(\frac{\cos\omega}{\omega} - \frac{\cos\omega_0}{\omega_0}) + S(\omega) - S(\omega_0) \quad (72)$$

Similarly, $I_2 = -\lambda \int_{\omega_0}^{\omega} \frac{\sin\omega}{\omega^2} d\omega$

$$I_2 = \lambda\{(\frac{\sin\omega}{\omega} - \frac{\sin\omega_0}{\omega_0}) - C(\omega) + C(\omega_0) \quad (73)$$

where $S(\omega) = \int \frac{\sin\omega}{\omega} d\omega$ and $C(\omega) = \int \frac{\cos\omega}{\omega} d\omega$ (74)

Now expressing sinω and cosω in infinite series, the integrals (74) yield,

$$S(\omega) = \omega - \frac{\omega^3}{3!3} + \frac{\omega^5}{5!5} - \frac{\omega^7}{7!7} + \cdots. \quad (75)$$

$$C(\omega) = \log\omega - \frac{\omega^2}{2!2} + \frac{\omega^4}{4!4} - \frac{\omega^6}{6!6} + \cdots. \quad (76)$$

$(2!\,2 = 2$multiplied by factorial $2)$

The curved path s described by the space vehicle in time t is given by,

$$\frac{ds}{dt} = v = v_0 e^{\alpha t} \quad (77)$$

Initially at t = 0, s = 0 so that,

$$s = v_0 \int_0^t e^{\alpha t} dt = \frac{v_0}{\alpha}(e^{\alpha t} - 1) \quad (78)$$

$$\frac{ds}{dt} = v\frac{dv}{ds} = \alpha v, \text{ Or, } v - v_0 = \alpha s \quad (79)$$

However, the time-varying thrust is obtained by using the relevant equations.

To maintain this horizontal flight the time-control of the required angle of attack ie lift coefficient has to be carried out in consequence of (59) and (64):

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$$C_L(t) = \frac{2gm(t)}{v_0^2 e^{2\alpha t} \rho S \cos\mu}$$

VII. Horizontal Accelerated Motion of Spacecraft /Aircraft with Parabolic Drag Leading to Variable Thrust Program

With parabolic polar drag defined[1] as,

$$D_P = K_D v^2 + \frac{km^2}{v^2} \quad (80)$$

where k is a constant involving the coefficient of induced drag ,choice of a variable thrust depends upon the control aspect of the rocket mass variation with time due to mode of the propellant consumption. In this context rectilinear motion of the spacecraft takes place with constant kinetic energy such that square of the velocity is i-

-nversely proportional to its instantaneous mass m(t) :

$$v^2 = \frac{1}{\delta m} \ (\delta = \text{constant}) \tag{81}$$

which reveals that as the mass decreases due to propellant consumption, the velocity increases.

$$\frac{dv}{dt} = -\frac{1}{2\sqrt{\delta} m^{\frac{3}{2}}} \frac{dm}{dt} \tag{82}$$

The main equation of motion is written using the above equations :

$$m\frac{dv}{dt} = T - D_P$$

$$\text{Or, } \frac{-1}{2}\sqrt{\frac{1}{m\delta}}\frac{dm}{dt} = -V_E \frac{dm}{dt} - (K_D \frac{1}{\delta m} + k\delta m^3)$$

$$\text{Or, } m\delta(\frac{1}{2}\sqrt{\frac{1}{m\delta}} - V_E)\frac{dm}{dt} = K_D + km^4\delta^2$$

$$\text{Or, } dt = \frac{1}{2k\delta^{\frac{3}{2}}}\frac{\sqrt{m}\,dm}{(C^8+m^4)} - \frac{V_E\, m dm}{k\delta(C^8+m^4)} \tag{83}$$

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$$\text{where } \frac{K_D}{k\delta^2} = C^8 \tag{84}$$

Resorting to the same initial conditions as earlier, the time−mass relation is given by,

$$t = \frac{1}{2k\delta^{\frac{3}{2}}} I_1(m) - \frac{V_E}{k\delta} I_2(m) \tag{85}$$

where

$$I_2(m) = -\int_{m_0}^{m} \frac{mdm}{(C^8+m^4)} = \frac{1}{2C^4}(\tan^{-1}\frac{m_0^2}{C^4} - \tan^{-1}\frac{m^2}{C^4}) \tag{86}$$

$$I_1(m) = \int_{m_0}^{m} \frac{\sqrt{m}\,dm}{(C^8+m^4)} \tag{87}$$

To evaluate this integral let us put m = c^2n^2 so that dm = $2c^2ndn$. Then (87) reduces to the form,

$$I_1(m) = \frac{2}{c^5}[j_1]_{Cn=\sqrt{m_0}}^{Cn=\sqrt{m}} \tag{88}$$

$$\text{where } j_1 = \int \frac{n^2}{1+n^8} dn = \int \frac{n^2}{(1+n^4)^2 - (\sqrt{2}n^2)^2} dn$$

$$= \frac{1}{2\sqrt{2}}\int \left(\frac{1}{n^4-\sqrt{2}n^2+1} - \frac{1}{n^4+\sqrt{2}n^2+1}\right) dn = \frac{1}{2\sqrt{2}}\{\int \frac{dn/n^2}{(n+\frac{1}{n})^2-(\sqrt{2+\sqrt{2}})^2} - \int \frac{dn/n^2}{(n+\frac{1}{n})^2-(\sqrt{2-\sqrt{2}})^2}\}$$

$$= \frac{1}{2\sqrt{2}}[\frac{1}{2}\{\int \frac{d(n-\frac{1}{n})}{(n-\frac{1}{n})^2+(\sqrt{2-\sqrt{2}})^2} - \int \frac{d(n+\frac{1}{n})}{(n+\frac{1}{n})^2-(\sqrt{2+\sqrt{2}})^2}\}$$

$$+\int \frac{1}{2}\{\frac{d(n+\frac{1}{n})}{(n+\frac{1}{n})^2-(\sqrt{2-\sqrt{2}})^2} - \int \frac{d(n-\frac{1}{n})}{(n-\frac{1}{n})^2+(\sqrt{2+\sqrt{2}})^2}\}]$$

It can be derived that $(n+\frac{1}{n})^2 - (\sqrt{2+\sqrt{2}})^2 = (n-\frac{1}{n})^2 + (\sqrt{2-\sqrt{2}})^2$.

$$(n+\frac{1}{n})^2-(\sqrt{2-\sqrt{2}})^2=(n-\frac{1}{n})^2+(\sqrt{2+\sqrt{2}})^2 \quad (89)$$

$$j_1=\frac{1}{8\sqrt{2}}[\frac{1}{\sqrt{2+\sqrt{2}}}\log\frac{n^2+\sqrt{2+\sqrt{2}}n+1}{n^2-\sqrt{2+\sqrt{2}}n+1}+\frac{1}{\sqrt{2-\sqrt{2}}}\log\frac{n^2-\sqrt{2-\sqrt{2}}n+1}{n^2+\sqrt{2-\sqrt{2}}n+1}$$

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$$+\frac{2}{\sqrt{2-\sqrt{2}}}\tan^{-1}\frac{n^2-1}{n}/\sqrt{2-\sqrt{2}}\ -\frac{2}{\sqrt{2+\sqrt{2}}}\tan^{-1}\frac{n^2-1}{n}/\sqrt{2+\sqrt{2}}\,] \quad (90)$$

The differential equation of heading by use of preceding equation (18) and (83) is obtained as

$$d\theta=\frac{g\tan\mu}{v}dt=(g\tan\mu)\sqrt{m\delta}\{\frac{1}{2k\delta^{\frac{3}{2}}}\frac{\sqrt{m}dm}{C^8+m^4}-\frac{V_E mdm}{k\delta(C^8+m^4)}\}$$

$$\text{Or, }\theta=\frac{g\tan\mu}{2k}\{\frac{1}{\delta}I_2(m)-\frac{2}{\delta^{\frac{1}{2}}}V_E I_3(m)\} \quad (91)$$

where $I_2(m)$ is derived in the previous text and $I_3(m)=\int_{m_0}^{m}\frac{m\sqrt{m}dm}{C^8+m^4}$ (92)

Putting $m = c^2/n^2$ so that $dm = -2c^2/n^3$ (92) becomes

$$I_3(m)=\frac{-2}{c^3}[j_1]_{n=\frac{c}{\sqrt{m_0}}}^{n=\frac{c}{\sqrt{m}}} \quad (93)$$

j_1 is obtained in the foregoing text.Hence recalling (83) and (91) we get with same initial conditions the horizontal distances at right angles traveled by the spacecraft from the starting point in time t :

$$x=\int_{m_0}^{m}\frac{1}{\sqrt{m\delta}}\cos\theta\,\frac{dt}{dm}dm$$

$$=\int_{m_0}^{m}\frac{1}{\sqrt{m\delta}}\cos[\frac{g\tan\mu}{2k}\{\frac{1}{\delta}I_2(m)-\frac{2}{\delta^{\frac{1}{2}}}V_E\sqrt{I_3(m)}\}]\{\frac{1}{2k\delta^{\frac{3}{2}}}\frac{\sqrt{m}}{C^8+m^4)}-\frac{V_E m}{k\delta(C^8+m^4)}\}dm \quad (94)$$

$$Y=\int_{m_0}^{m}\frac{1}{\sqrt{m\delta}}\sin[\frac{g\tan\mu}{2k}\{\frac{1}{\delta}I_2(m)-\frac{2}{\delta^{\frac{1}{2}}}V_E\sqrt{I_3(m)}\}]\left\{\frac{1}{2k\delta^{\frac{3}{2}}}\frac{\sqrt{m}}{C^8+m^4}-\frac{V_E m}{k\delta(C^8+m^4)}\right\}dm \quad (95)$$

The cumbersome integrals (94) and (95) can be determined numerically with given values of the parameters. However the lift as a control parameter has to be balanced such that

$$L\cos\mu=\frac{1}{2}C_L\rho Sv^2\cos\mu=mg$$

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$$C_L=\frac{2\delta m^2 g}{\rho S\cos\mu} \quad (96)$$

whereas the mass and time t are related by equations (85) to (87),which though complicated, can help predict the mass of the spaceraft at time t by tabulation with numerical values. The curvilinear distance traveled by the rocket can be obtained by combining equations (81) and(83) and solving the result.

$$\frac{ds}{dt}=v=\sqrt{\frac{1}{m\delta}}\ \text{Or, } ds=\sqrt{\frac{1}{m\delta}}\frac{dt}{dm}dm$$

$$\text{Or, } ds=\sqrt{\frac{1}{m\delta}}\{\frac{1}{2k\delta^{\frac{3}{2}}}\frac{\sqrt{m}dm}{C^8+m^4}-\frac{V_E mdm}{k\delta(C^8+m^4)}\}$$

Or, ds= $\frac{1}{2k\delta^2}\frac{dm}{C^8+m^4} - \frac{V_E\sqrt{m}dm}{k\delta^{\frac{3}{2}}(C^8+m^4)}$

Or,s=$\int_{m_0}^{m}\{\frac{1}{2k\delta^2}\frac{dm}{C^8+m^4} - \frac{V_E\sqrt{m}dm}{k\delta^{\frac{3}{2}}(C^8+m^4)}\}=\frac{I_0(m)}{2k\delta^2} - \frac{V_E}{k\delta^{\frac{3}{2}}}I_1(m)$

where $I_1(m)$ is determined in the foregoing text.

$$I_0(m) = \int_{m_0}^{m}\frac{dm}{C^8+m^4} = \frac{1}{C^6}\int_{\frac{m_0}{c^2}}^{\frac{m}{c^2}}\frac{dn}{1+n^4}=\frac{1}{C^6}[j_0(n)]_{\frac{m_0}{c^2}}^{\frac{m}{c^2}} \quad (97)$$

$$j_0(n) = \frac{1}{2}\int\frac{d\left(n-\frac{1}{n}\right)-d\left(n+\frac{1}{n}\right)}{n^2+\frac{1}{n^2}};\ n^2+\frac{1}{n^2} = \left(n+\frac{1}{n}\right)^2-(\sqrt{2})^2 = \left(n-\frac{1}{n}\right)^2+(\sqrt{2})^2$$

so that $j_0(n) = \frac{1}{2}\int\frac{d\left(n-\frac{1}{n}\right)}{\left(n-\frac{1}{n}\right)^2+(\sqrt{2})^2} - \frac{1}{2}\int\frac{d\left(n+\frac{1}{n}\right)}{\left(n+\frac{1}{n}\right)^2-(\sqrt{2})^2}$

$$\text{Or, } j_0(n) = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{n^2-1}{n\sqrt{2}} + \frac{1}{4\sqrt{2}}\log\frac{n^2+\sqrt{2}n+1}{n^2-\sqrt{2}n+1} \quad (98)$$

VIII. Horizontal Flight with Constant Angle of Attack

Since vertical component of the lift balances weight of the rocket/aircraft, in the light of the foregoing contents, square of the velocity becomes proportional.

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To the instantaneous mass where the angle of attack, viz, lift coefficient is constant. Then

$$\frac{ds}{dt} = v = \sqrt{\sigma m},\ \sigma = \text{constant} \quad (99)$$

$$L\cos\mu = \frac{1}{2}C_L\rho Sv^2\cos\mu = mg$$

Combining this equation with (94), one gets

$$C_L = \frac{2g}{\sigma\rho S\cos\mu} = \text{constant} \quad (100)$$

Hence the associated thrust program suggests decelerated motion of the rocket. Consequently the parabolic polar drag is given by

$$D = D_{P'} = K_D m\sigma + \frac{k}{\sigma}m \quad (101)$$

which is used in the main equation of motion to yield the following equation

$m\frac{dv}{dt} = T - D_{P'}$ By use of (18) and (101)

$$\frac{1}{2}\sqrt{\sigma m}\frac{dm}{dt} = -V_E\frac{dm}{dt} - (K_D\sigma + \frac{k}{\sigma})m$$

$$\text{Or, } (\frac{1}{2}\sqrt{\sigma m} + V_E)\frac{dm}{dt} = -(K_D\sigma + \frac{k}{\sigma})m$$

$$\text{Or, } \frac{1}{2}\frac{\sigma^{\frac{3}{2}}}{\sqrt{m}}dm + \frac{V_E\sigma dm}{m} = -(K_D\sigma^2 + k)dt \quad (102)$$

Integrating (102) subject to the same initial conditions as earlier, we have

$$V_E\sigma\log\frac{m_0}{m} + \sigma^{\frac{3}{2}}(\sqrt{m_0} - \sqrt{m}\,) = (K_D\sigma^2 + k)t \tag{103}$$

which gives the mass variation law for this thrust program that depends on the propellant mass flow. The thrust is evaluated from (102) as a function of mass m:

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$$T = -V_E\frac{dm}{dt} = V_E\frac{(K_D\sigma^2+k)}{\frac{1}{2}\frac{\sigma^{\frac{3}{2}}}{\sqrt{m}}+\sigma\frac{V_E}{m}} \tag{104}$$

Now for heading, $d\theta = \frac{g\tan\mu}{v}\frac{dt}{dm}dm = -\frac{g\tan\mu}{\sqrt{\sigma m}}\frac{\frac{1}{2}\frac{\sigma^{\frac{3}{2}}}{\sqrt{m}}dm+\frac{V_E\sigma dm}{m}}{(K_D\sigma^2+k)}$. Hence he heading θ(t) at time t is obtained with the help of the initial conditions (20):

$$\theta = \frac{g\tan\mu}{K_D\sigma^2+k}\{\frac{\sigma}{2}\log\frac{m_0}{m}) + 2V_E\sqrt{\sigma}\,(\sqrt{\frac{1}{m}} - \sqrt{\frac{1}{m_0}})\} \tag{105)}$$

The horizontal distances at right angles are given by,

$$x = \int_{m_0}^{m} v\cos\theta\,\frac{dt}{dm}\,dm$$

$$= \int_{m}^{m_0}\frac{\sqrt{m\sigma}}{K_D\sigma^2+k}[\cos\{\frac{g\tan\mu}{K_D\sigma^2+k}\{\frac{\sigma}{2}\log\frac{m_0}{m}) + 2V_E\sqrt{\sigma}\,(\sqrt{\frac{1}{m}} - \sqrt{\frac{1}{m_0}})\}]\,(\frac{\sigma^{\frac{3}{2}}}{2\sqrt{m}} + \frac{V_E\sigma}{m})dm \tag{106}$$

The expression for y is acquired by replacing 'cos' by 'sin' in (106). Finally the curvilinear distance s described at time t is given by.

$$\frac{ds}{dt} = v = \sqrt{m\sigma}$$

$$\text{Or, } ds = \sqrt{m\sigma}\frac{dt}{dm}dm = -\sqrt{m\sigma}\frac{\frac{1}{2}\frac{\sigma^{\frac{3}{2}}}{\sqrt{m}}+\frac{V_E\sigma}{m}}{(K_D\sigma^2+k)}dm \tag{107}$$

Integrating (107) using the initial conditions: at t = 0, s = 0,

$$s = \frac{1}{K_D\sigma^2+k}\{\frac{\sigma^2}{2}(m_0 - m) + 2V_E\sigma^{\frac{3}{2}}(\sqrt{m_0} - \sqrt{m})\} \tag{108}$$

Now let us find mass m as an explicit function of distances. Denoting $\frac{\sigma^2}{2(K_D\sigma^2+k)}$ and $\frac{2V_E\sigma^{\frac{3}{2}}}{K_D\sigma^2+k}$ by p and 2q respectively, equation(108) becomes,

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$$s = p(m_0 - m) + 2q(\sqrt{m_0} - \sqrt{m})$$

$$= p(\sqrt{m_0} - \sqrt{m})(\sqrt{m_0} + \sqrt{m}) + 2q(\sqrt{m_0} - \sqrt{m})\} \tag{109}$$

Letting $n = (\sqrt{m_0} - \sqrt{m})$ in equation (109), it reduces to the form,

$$s = pn(2\sqrt{m_0} - n)) + 2qn$$

Or, $pn^2 - 2(p\sqrt{m_0} + q)n + s = 0$

Or, $n = \frac{p\sqrt{m_0}+q \pm \sqrt{(p\sqrt{m_0}+q)^2 - ps}}{p}$

Or, $\sqrt{m} = \sqrt{m_0} - n = \frac{-q + \sqrt{(p\sqrt{m_0}+q)^2 - ps}}{p}$ (110)

Or, $m = \left(\frac{-q + \sqrt{(p\sqrt{m_0}+q)^2 - ps}}{p}\right)^2 < m_0$ (111)

$ps < (p\sqrt{m_0} + q)^2$, Binomial expansion in (110)neglecting higher terms yields.

$$\sqrt{m} = \frac{-q + (p\sqrt{m_0}+q)\left(1 - \frac{ps}{2(p\sqrt{m_0}+q)^2} + \cdots\right)}{p} = \frac{p\sqrt{m_0} - \frac{ps}{2(p\sqrt{m_0}+q)}}{p}$$

Or, $\sqrt{m} = \sqrt{m_0} - \frac{s}{2(p\sqrt{m_0}+q)}$

Or, $m = \left(\sqrt{m_0} - \frac{s}{2(p\sqrt{m_0}+q)}\right)^2$ (112)

Finally substituting the values of p and q from the text above, equation (110) gives,

$$m = \left(\sqrt{m_0} - \frac{s}{\left(\frac{\sigma^2}{(K_D\sigma^2+k)}\right)\sqrt{m_0} + \frac{2V_E\sigma^{\frac{3}{2}}}{K_D\sigma^2+k}}\right)^2 \quad (113)$$

Hence the propellant consumption at the end of time t is given by opening the square (113):

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$$\rho = m_0 - m = \frac{s}{\left(\frac{\sigma^2}{(K_D\sigma^2+k)}\right)\sqrt{m_0} + \frac{2V_E\sigma^{\frac{3}{2}}}{K_D\sigma^2+k}}\left(2\sqrt{m_0} - \frac{s}{\left(\frac{\sigma^2}{(K_D\sigma^2+k)}\right)\sqrt{m_0} + \frac{2V_E\sigma^{\frac{3}{2}}}{K_D\sigma^2+k}}\right) \quad (114)$$

However the propellant consumption without any approximation is obtained due to relation (111):

$$\rho' = m_0 - m = m_0 - \left(\frac{-q + \sqrt{(p\sqrt{m_0}+q)^2 - ps}}{p}\right)^2 \quad (115)$$

IX. Conclusion

In this design various thrust programs are chosen in such a way that acceleration, velocity and distance traveled by the spacecraft can be evaluated more or less as explicit functions of its instantaneous mass owing to the propellant consumption entailing a specific mass variation law. Varieties of rocket performance can be studied in accordance with various realistic mass-variation laws.

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