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A Study of P_1-Curvature Tensor in LP-Sasakian Manifolds

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Abstract **– In this paper, we study flatness, symmetric and semi-symmetric properties of the P_1-curvature tensor in a LP-Sasakian manifold. We further study the recurrent relations. 2010 Mathematics subject Classification: 53C15, 53C40.**

Keywords **– Flatness, Semi-Symmetric, Symmetric, Recurrent.**

I. Introduction

An n-dimensional real differentiable manifold M_n is said to be Lorentzian para (LP)-Sasakian if it admits a (1, 1) tensor ϕ, a C^∞ vector field ξ, a C^∞1-form η and a Lorentzian metric g which satisfies Mishra [2].

$$\eta(\xi) = -1, g(X,\xi) = \eta(X) \tag{1.1}$$

$$\phi^2(X) = X + \eta(X)\xi \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.3}$$

$$D_X\xi = \phi X \tag{1.4}$$

$$(D_X\phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{1.5}$$

where D_X denotes the covariant differentiation with respect to g, and X and Y are any arbitrary vector fields on M_n .

In an LP-Sasakian manifold M_n with structure (ϕ, ξ, η, g), it can be seen [4] that,

$$\phi(\xi) = 0, \eta(\phi X) = 0 \tag{1.6}$$

$$rank\ (\phi) = n - 1 \tag{1.7}$$

If we put,

$$\phi'(X,Y) = g(X,Y) \tag{1.8}$$

Then the tensor $\phi'(X,Y)$ is symmetric in X and Y. Thus, we have $\phi'(X,Y) = \phi'(Y,X)$.

In an n-dimensional LP-Sasakian manifold with the structure (ϕ, ξ, η, g) we have,

$$R'(X,Y,Z,U) = g(R(X,Y)Z,U) = g(\{g(Y,Z)X - g(X,Z)Y\},U) = g(Y,Z)g(X,U) - g(X,Z)g(Y,U). \tag{1.9}$$

Again, putting $U = \xi$, relation (1.9) becomes,

$$R'(X,Y,Z,\xi) = g(Y,Z)\, g(X,\xi) - g(X,Z)\, g(Y,\xi) \tag{1.10}$$

Using (1.1), relation (1.9) yields,

$$R'(X,Y,Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) = \eta(R(X,Y)Z) \Leftrightarrow g(R(X,Y)Z,\xi) = eta(R(X,Y)Z, \tag{1.11}$$

where $g(X,Y)Z$ is the metric tensor representing potential

$$S(X,Y) = Ric(X,Y) = g(\phi X,Y) = (n-1)g(X,Y) \tag{1.12}$$

is the Ricci tensor representing the metric tensor,

$$S(\xi,\xi) = (n-1)g(\xi,\xi) = (n-1)\eta(\xi) = -(n-1), \tag{1.13}$$

R' is the (0,4) curvature tensor, and $S(X,Y) = Ric(X,Y)$ is the Ricci tensor.

II. P_1-Curvature Tensor in LP-Sasakian Manifolds

Pokhariyal [3] gave the definition of W_3-curvature tensor as $W_3(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)$ $Ric[(X,T) - g(Y,T)Ric(X,Z)]$ and Pokhariyal [3] defined the P_1-curvature tensor as $P_1(X,Y,Z,T) = \frac{1}{2}[W_3(X,Y,Z,T) - W_3(Y,X,Z,T)] = R(X,Y,Z,T) + \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T) - g(Y,T)\,Ric(X,Z) - g(X,\ Z)$ $Ric(Y,T) + g(X,T)Ric(Y,Z)$

Definition 2.1

An LP-Sasakian manifold M_n is said to be flat if the Riemannian curvature tensor vanishes identically, that is $R(X,Y)Z = 0$.

Definition 2.2

An LP-Sasakian manifold M_n is said to be P_1-flat if the P_1-curvatiure tensor vanishes identically, that is $P_1(X,Y)Z = 0$.

Theorem 2.3

A P_1-flat LP-Sasakian manifold is an Einstein manifold.

Proof

If $P_1(X,Y,Z,T) = 0$ in the above definition of P_1-Curvature Tensor, we have $0 = R(X,\ Y,\ Z,\ T) + \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z) - g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z)$.

Using $Ric(X,Y) = (n-1)g(X,Y)$, we have $R(X,Y,Z,T) = \frac{(n-1)}{2(n-1)}[-g(X,T)g(Y,Z) + g(Y,T)g(X,Z) + g(X,Z)g(Y,T) - g(X,T)g(Y,Z) = g(Y,T)g(X,Z) - g(X,T)g(Y,Z)$.

Next, when we contract with respect to X and Y, we have $S(Y,T) = R(X,Y,Z,T) = g(Y,T)g(X,Z) - g(\xi,T)g(Y,\xi) = ng(Y,T) - g(Y,T) = (n-1)g(Y,T)$ and thus, we have the Theorem.

III. P_1-Semi-Symmetric LP-Sasakian Manifolds

Definition 3.1

An LP-Sasakian manifold is said to be semi-symmetric if $R(X,Y)R(Z,U)V = 0$.

Definition 3.2

An LP-Sasakian manifold is called P_1 semi-symmetric if $R(X,Y)P_1(Z,U)V = 0$.

Theorem 3.3

A P_1 semi-symmetric LP-Sasakian manifold is P_1-flat.

Proof

When we take the inner product with respect to ξ, we have $g(R(X,Y)P_1(Z,U)V,\xi) = R(X,Y,P_1(Z,U)V,\xi) = g(X,\xi)g(Y,P_1(Z,U)V - g(Y,\xi)g(X,P_1(Z,U)V) = \eta(X)P_1(Z,U,V,Y) - \eta(Y)P_1(Z,U,V,X) = 0.$

Since $\eta(X) \neq 0$ *and* $\eta(Y) \neq 0$, we conclude that $P_1(Z,U,V,Y) = 0$ and $P_1(Z,U,V,X) = 0$ and thus follows the theorem.

IV. P_1-SYMMETRIC LP-SASAKIAN MANIFOLDS

Definition 4.1

An LP-Sasakian manifold is said to be P_1 symmetric if,

$$\nabla_U P_1(X,Y)Z = P_1'(U,X,Y)Z = 0 \quad (4.1)$$

Theorem 4.2

A P_1 symmetric and P_1 semi-symmetric LP-Sasakian manifold M is P_1-flat.

Proof.

In Theorem 3.3, we found out that a P_1 semi-symmetric LP-Sasakian manifold P_1-flat manifold. If an LP-sasakian space is P_1-symmetric, this implies that.

$$\nabla_U P_1(X,Y)Z = R(X,Y)P_1(Z,U)V - P_1(R(X,Y)Z,U)V - P_1(Z,R(X,Y)U,V) - P_1(Z,U)R(X,Y)V = 0 \quad (4.2)$$

Next, we take the inner product of (4.2) with respect to ξ to get,

$$g(R(X,Y)P_1(Z,U)V,\xi) - g(P_1(R(X,Y)Z,U)V,\xi) - g(P_1(Z,R(X,Y)U)V,\xi) - g(P_1(Z,U)R(X,Y)V,\xi) = 0$$

Or

$$R'(X,Y,P_1(Z,U)V,\xi) - P_1'(R(X,Y)Z,U,V,\xi) - P_1'(Z,R(X,Y)U,V,\xi) - P_1'(Z,U,R(X,Y)V,\xi) = 0$$

Next, we compute each of the above expressions separately to get,

$$R'(X,\ Y,\ P_1(Z,U)V,\xi) = g(X,\xi)g(Y,P_1(Z,U)V) - g(Y,\xi)g(X,P_1(Z,U)V = \eta(X)P_1'(Y,Z,U,V) - \eta(Y)P_1'(X,Z,U,V) \quad (4.3)$$

$$P_1'(R(X,\ Y)Z,\ U,\ V,\xi) = P_1'(g(Y,Z)X - g(X,Z)Y,U,V,\xi) = P_1'(g(Y,Z)X,U,V,\xi) - P_1'(g(X,Z)Y,U,V,\xi) = g(Y,Z)P_1'(X,U,V,\xi) - g(X,Z)P_1'(Y,U,V,\xi) \quad (4.4)$$

$$P_1'(Z,R(X,Y)U,V,\xi) = P_1'(Z,g(Y,U)X - g(X,U)Y,V,\xi) = g(Y,U)P_1'(Z,X,V,\xi) - g(X,U)P_1'(Z,Y,V,\xi) \quad (4.5)$$

$$P_1'(Z,U,R(X,Y)V,\xi) = P_1'(Z,U,g(Y,U)X - g(X,V)Y,\xi) = g(Y,V)P_1'(Z,U,X,\xi) - g(X,V)P_1'(Z,U,Y,\xi) \quad (4.6)$$

Next, we put together (4.3), (4.4), (4.5) and (4.6), we get, $\eta(X)P_1'(Y,Z,U,V) - \eta(Y)P_1'(X,Z,U,V) + g(Y,Z)\,P_1'(X,U,V,\xi) - g(X,Z)\,P_1'(Y,U,V,\xi) + g(Y,U)P_1'(Z,X,V,\xi) - g(X,U)\,P_1'(Z,Y,V,\xi) + g(Y,V)\,P_1'(Z,U,X,\xi) - g(X,V)P_1'(Z,U,Y,\xi) = 0.$

Since $\eta(X) \neq 0$ *and* $\eta(Y) \neq 0$ and since $g(Y,Z) \neq g(X,Z) \neq g(Y,U) \neq g(X,U) \neq 0$ this implies that $p_1'(Y,Z,U,V) = 0 = g(Y, P_1(Z,U)V = \nabla_Y P_1(Z,U)V$ for all since $X,Y,Z,U,V \in T(M)$, which implies that $P_1(Z,U)V = 0$ and thus, follows the theorem.

V. P_1-Recurrent LP-Sasakian Manifolds

In this section, we study some of the geometric properties of the p_1-Curvature Tensor which is recurrent on an LP-Sasakian manifold *M*.

Pokhariyal [5] defined, $\nabla_U P_1(X,Y)Z = B(U)P_1(X,Y)Z$.

The notation of Ricci-recurrent is defined similarly as $\nabla_U Ric(X,Y) = B(U)Ric(X,Y)$.

Theorem 5.1

A P_1 symmetric, P_1 semi-symmetric and P_1 recurrent LP-Sasakian manifold is a P_1-flat manifold.

Proof.

If a LP-Sasakian space is P_1 symmetric, P_1 semi-symmetric and P_1 recurrent, then $\nabla_U P_1(X,Y,Z)V = B(U)P_1(X,Y,Z)V = R(X,Y)P_1(Z,V)U - P_1(R(X,Y)Z,U)V = 0$.

Which from the statements of the theorem and theorem (4.2), the claim is prove.

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Conflict of Interest

The authors declare that there is no conflict in publishing this paper.

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