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A New Iterative Method of Solution of Nonlinear Equations Derived from Simpson and Trapezoidal Quadrature Rules

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***Abstract* – Many problems in Engineering, Computer Science, and Applied Mathematics among other fields lead to the solution of nonlinear equations. Nontrivial cases of such equations can only be solved iteratively. Development of efficient and effective iterative methods therefore is very important, and can impact positively the task of finding numerical solutions of many real-world problems. This paper is devoted to developing a new and efficient iterative method for solving nonlinear equations. The new method is derived from a weighted combination of the Simpson and Trapezoidal schemes, which are derivatives of the Newton-Cotes quadrature rules of integration, in conjunction with a new iterative method reported in the literature. The newly developed method was shown to converge and its performance compared with Newton's and a newer method, using several benchmark problems, with their run-time, the number of iterations before converging, and the size of the error of the approximated solutions as the indicators. The Results showed that the newly developed method performed as good as the classical Newton's method and the leading method in some of the cases and better than them in many of the cases. This provides grounds for confidence in the newly developed method and provides prospects for further refinement in future work.**

***Keywords* – Quadrature Rules, Nonlinear Equations, Iterative Methods, Number of Iterations, Run-Time.**

I. Introduction

Nonlinear equations occur in the modelling of real problems in the fields of Science, Engineering and Applied Mathematics, as in many other fields, and invariably are non-trivial problems for which analytical solutions are impossible (Khushbu et al., 2019; Li et al., 2015) and therefore require numerical methods of solution. The numerical methods are in general iterative in nature, and characteristically estimate approximate solutions for the equations in a step-by-step manner beginning with an initial approximate solution in which the next step is supposed to yield an improved approximate solution, and so on. The process continues until the method converges, or until a certain tolerance is achieved. Convergence is therefore critical for any iterative method. Furthermore, the speed of convergence is also very important (Khushbu et al., 2019), as that can have practical, or even economic, consequence, especially for real problems. Technically, the solution found is called a root of the equation. Depending on the nature of the equation there can be a single real distinct root or many real distinct or repeated roots, and some can be imaginary as well (Qureshi, 2018). The focus of this research is in finding real and distinct (or simple) roots.

Estimating roots of nonlinear equations has attracted researchers and practitioners for many years. Many variants of accelerated methods that proved instrumental in estimating the roots of nonlinear equations have been introduced by researchers. Many iterative methods have been developed and modified by several researchers. Some of these methods are the Bisection, Regula-Falsi, and Newton methods (Sehrish et al., 2020), which have the same or better performance. While many numerical solution methods which involve the use of derivatives have been developed, many other derivative-free iterative methods with better rates of convergence have also been developed (Khushbu et al., 2019).

Finding efficient numerical methods for solving nonlinear equations is essential in view of their practical implications (Azure et al., 2020), which includes providing answers to many real-world problems. It is no wonder, therefore, that researchers pay much attention to developing new theories and methods which are more efficient at solving nonlinear equations (Kou et al., 2007). The convergence and performance characteristics of some classical methods are highly sensitive to the initial guess of the solution. Example is the Newton's method (Azure et al., 2021a). However, for the majority of nonlinear equations, it is quite challenging to choose a good initial solution (Azure et al., 2021b; Reis et al., 2017). In order to solve nonlinear equations, a variety of traditional numerical methods and clever iterative methods are used, which can handle the issue of choosing an appropriate starting solution (Luo et al., 2008; Mo et al., 2009).

Quadrature rules of numerical integration have recently become popular avenues for finding efficient solution algorithms for nonlinear equations. One of the early works in this area was by Frantini and Sormani (2003), who used the rectangular and the trapezoidal rules to derive an iterative solution method for nonlinear equations.

Mir et al. (2010) proposed a quadrature based-three step iterative method for nonlinear equations. They used the rectangular rule and the Newton's method to derive their algorithm which was said to have a convergence order of eight. Ababneh (2012) proposed a modification to the Newton's method using the Trapezoidal Quadrature rule and the Contra Harmonic mean to derive a new method that had cubic convergence; and examples showed it could compete with the classical Newton's method.

Eskandari (2017) proposed a method using the Simpson's integration and used to modify the Newton's iteration formula for solving nonlinear equations. Singh and Singh (2017) used the Trapezoidal rule and a harmonic mean to modify the Newton's method for solving nonlinear equations. The proposed method was seen to be more efficient when compared with the Newton's method. Furthermore, Ahmad and Singh (2017) modified the Newton's method in their work. They used the Mid-point and the Simpson 1/3 rules to develop an iterative method for solving nonlinear equations. The convergence of the developed iterative method was of order 3, and when compared with existing single step iterative methods, it was found to provide best results.

Masjed-Jamei & Moalemi (2018) modified the Newton's method using the Gauss-Legendre quadrature rule. The proposed method was compared to the Classical Newton's method, the Arithmetic mean Newton's method, the Harmonic mean Newton's method and the Geometric mean Newton's method. The results showed that the proposed method was superior. Thunkral (2018) used the Simpson's quadrature rule to develop an iterative method for solving nonlinear equations. The proposed Simpson fifth-order method was compared with the third-order and the fourth-order Simpson-type iterative methods from which he concluded that the proposed method had a better order of convergence. Also, Qureshi et al., (2018) proposed an iterative method for solving nonlinear equations. In this paper, the Newton's method was modified using the Midpoint and the Trapezoidal quadrature rules. The proposed method was compared with the Newton's method and was found to be better than it. Also, Qureshi (2018) used the Simpson quadrature rule to modify the Newton's method. The proposed iterative method was a third-order method and was observed to be pragmatic and faster on few examples of nonlinear equations when compared with a variant of Newton's methods.

Sehrish et al. (2020) used the Midpoint rule and the Trapezoidal rule to propose an algorithm for solving nonlinear equations. The proposed method is a two-step scheme with third order convergence. The superiority of the proposed method was demonstrated over the others, especially its main feature of cost efficiency, since it used lesser evaluations to achieve similar accuracy when compared with the other methods. Sana et al. (2020) proposed a new iterative method for solving nonlinear equations. The Midpoint rule, the Trapezoidal rule and the Decomposition technique were used to develop a new iterative method. The proposed method was compared with well-known third and fourth order convergent iterative methods and was observed to be better than them.

Srivastava et al. (2021) proposed a new three-step Newton method for solving a system of nonlinear equations. They used the Gauss Quadrature rule to derive the proposed method which had a sixth order convergence rate. The new method was compared to other methods and was found to give the best results. Azure et al. (2021b) used a weighted combination of the trapezoidal, midpoint, the Simpson $1/3$ and Simpson $3/8$ Quadrature rules to propose a new iterative method for solution of nonlinear systems of equations. The weighted combination of the midpoint and Simpson $3/8$ was observed to outperform the other Broyden-like methods when compared using some given examples.

From the above reviews, it is quite evident that Quadrature rules such as the Midpoint, Trapezoidal, Simpson's, Simpson's $1/3$ and Simpson's $3/8$ have been very successful in the process of evolving new and efficient methods for solving nonlinear equations. Also, Gauss Quadrature and Gauss-Legendre Quadrature rules have also performed well in the race to developing new iterative methods and therefore have the potential for deriving iterative solution methods for nonlinear equations. In the next section an overview of the Trapezoidal and Simpson's quadrature rules is briefly covered. Subsequently, in the next section, the new iteration method is developed, followed by a section devoted to testing the new method against two other known methods. Afterwards, the results are discussed in another section and the paper concluded in the last section.

II. Some Quadrature Rules of Numerical Integration

Well-known quadrature rules of integration which provide the bases for the development of new iteration methods for soling nonlinear equations are derivatives of the Newton-Cotes quadrature rule of integration. In this section attention is devoted to the Trapezoidal, and the Simpson rules. The Newton-Cotes quadrature rule of integration of form $\int_a^b f'(x)$, where $f'(x)$ denotes derivative of the function $f(x)$ is in general given by:

$$I = \int_{x_0}^{x_0+nh} f'(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 - \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{255n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots\right] \quad (2.1)$$

where $h = \frac{b-a}{n}$, $n = 1, 2, 3, \dots$; Δ denotes change in y_0, and $y_0 = f(x_0)$. From the Newton-Cotes quadrature formula (2.1), by taking $n = 1$ and setting the Newton-Cotes quadrature to 1^{st} order results in the Trapezoidal rule as follows:

$$\int_a^b f'(x)dx = h[f'(a) + \tfrac{1}{2}(f'(b) - f'(a))] = h\left[f'^{(a)} - \tfrac{1}{2}f'^{(a)} + \tfrac{1}{2}f'^{(b)}\right] = h[\tfrac{1}{2}f'(a) + \tfrac{1}{2}f'(b)].$$

But $h = b - a$, therefore:

$$\int_a^b f'(x)dx \approx \frac{(b-a)}{2}(f'(a) + f'(b)) \quad (2.2)$$

From the Newton-Cotes quadrature formula (2.1), by taking $n = 3$ and setting the Newton-Cotes quadrature to 3^{nd} order while taking the polynomial through the a, $\frac{2a+b}{3}$, $\frac{a+2b}{3}$ and b. this result in the formula:

$$\int_a^b f'(x)dx \approx \frac{3h}{8}\left(f'(a) + 3f'\left(\frac{2a+b}{2}\right) + 3f'\left(\frac{a+2b}{2}\right) + f'(b)\right)$$

Since $3h = b - a$, this implies that;

$$\int_a^b f'(x)dx \approx \frac{b-a}{8}\left(f'(a) + 3f'\left(\frac{2a+b}{2}\right) + 3f'\left(\frac{a+2b}{2}\right) + f'\ (b)\right) \tag{2.3}$$

which is the Simpson 3/8 quadrature rule.

III. Derivation of the Proposed Iterative Method

Suppose that $f(t)$ is an equation that is sufficiently differentiable over a closed interval $[x_n, x]$ of the real line. Also, supposed that $f(x) = 0$ is a continuous single variable nonlinear equation that is sufficiently differentiable over the given interval, $[x_n, x]$, with $f'(x) \neq 0$. Then, the trapezoidal quadrature rule on the range x_n to x is given as:

$$\int_{x_n}^x f'(t)dt \approx \frac{(x-x_n)}{2}[f'(x_n) + f'(x)] \tag{3.1}$$

Also, the Simpson 3/8 quadrature rule on the range x_n to x is given as:

$$\int_{x_n}^x f'(t)dt \approx \frac{(x-x_n)}{8}\left[f'(x_n) + 3f'\left(\frac{2x_n+x}{3}\right) + 3f'\left(\frac{x_n+2x}{3}\right) + f'(x)\right] \tag{3.2}$$

Applying a weight of 0.5 on (3.1) and (3.2) respectively and summing yields:

$$\int_{x_n}^x f'(t)dt = \frac{(x-x_n)}{4}[f'(x_n) + f'(x)] + \frac{(x-x_n)}{16}\left[f'^{(x_n)} + 3f'^{\left(\frac{2x_n+x}{3}\right)} + 3f'^{\left(\frac{x_n+2x}{3}\right)} + f'(x)\right] \tag{3.3}$$

Since $\int_{x_n}^x f'(t)dt = f(x) - f(x_n)$, therefore:

$$f(x) - f(x_n) =$$

$$\frac{(x-x_n)}{4}[f'(x_n) + f'(x)] + \frac{(x-x_n)}{16}\left[f'(x_n) + 3f'\left(\frac{2x_n+x}{3}\right) + 3f'\left(\frac{x_n+2x}{3}\right) + f'(x)\right] \tag{3.4}$$

And since $f(x) = 0$, the left-hand side of (3.4) becomes:

$$-f(x_n) = \frac{(x-x_n)}{4}[f'(x_n) + f'(x)] + \frac{(x-x_n)}{16}\left[f'(x_n) + 3f'\left(\frac{2x_n+x}{3}\right) + 3f'\left(\frac{x_n+2x}{3}\right) + f'(x)\right] \tag{3.5}$$

Multiplying both sides of (3.5) by $\frac{16}{(x-x_n)}$ yields:

$$\frac{-16f(x_n)}{(x-x_n)} = 4f'(x_n) + 4f'(x) + f'(x_n) + 3f'\left(\frac{2x_n+x}{3}\right) + 3f'\left(\frac{x_n+2x}{3}\right) + f'(x)\,\frac{-16f(x_n)}{(x-x_n)} = 5f'(x_n) +$$
$$3f'\left(\frac{2x_n+x}{3}\right) + 3f'\left(\frac{x_n+2x}{3}\right) + 5f'(x) \tag{3.6}$$

Multiplying both sides of (3.6) by $(x - x_n)$ and making x the subject yields:

$$x = x_n - \frac{16f(x_n)}{\left[5f'(x_n)+3f'\left(\frac{2x_n+x}{3}\right)+3f'\left(\frac{x_n+2x}{3}\right)+5f'(x)\right]} \tag{3.7}$$

Taking x as x_{n+1} produces the iterative scheme:

$$x_{n+1} = x_n - \frac{16f(x_n)}{\left[5f'(x_n)+3f'\left(\frac{2x_n+x}{3}\right)+3f'\left(\frac{x_n+2x}{3}\right)+5f'(x)\right]} \tag{3.8}$$

By the Newtons iteration method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$; replacing x in the functions in (3.8) by the right-hand side of the Newton method gives:

$$x_{n+1} = x_n - \frac{16f(x_n)}{\left[5f'(x_n) + 3f'\left(\frac{2x_n + x_n - \frac{f(x_n)}{f'(x_n)}}{3}\right) + 3f'\left(\frac{x_n + 2\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{3}\right) + 5f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]}$$

$$x_{n+1} = x_n - \frac{16f(x_n)}{\left[5f'(x_n)+3f'\left(x_n-\frac{f(x_n)}{3f'(x_n)}\right)+3f'\left(x_n-\frac{2f(x_n)}{3f'(x_n)}\right)+5f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]} \tag{3.9}$$

The resulting expression (3.9) is a proposed new iterative method for solution of nonlinear equations. The next section assesses the error and convergence characteristics of the proposed new method.

IV. Error and Convergence of the New Method

Suppose that $\alpha \in (a, b)$ is a simple real root of a sufficiently differentiable function $f: (a, b) \subseteq R \to R$ on an open interval (a, b), with $f(x) = 0$, on the interval, $f'(x) \neq 0$ on the interval, and $x_0 \in (a, b)$ is chosen sufficiently closed to α. Then the global error, e_{n+1}, of the approximated solution, x_{n+1}, is given by $e_{n+1} = x_{n+1} - \alpha$, where n is the number of iterations. The absolute general error, $|e_{n+1}|$, of the approximated solution, x_{n+1}, is given by $|e_{n+1}| = |x_{n+1} - \alpha|$. Therefore (from (3.9)):

$$|e_{n+1}| = \left|x_n - \frac{16f(x_n)}{\left[5f'(x_n)+3f'\left(x_n-\frac{f(x_n)}{3f'(x_n)}\right)+3f'\left(x_n-\frac{2f(x_n)}{3f'(x_n)}\right)+5f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]} - \alpha\right| \tag{3.10}$$

If the scheme (3.9) is convergent, then limit of $|e_{n+1}|$ as $n \to \infty$, $x_n \to \alpha$. If that is the case, we have:

$$\lim_{n\to\infty}|e_{n+1}| = \left|\alpha - \frac{16f(\alpha)}{\left[5f'(\alpha)+3f'\left(\alpha-\frac{f(\alpha)}{3f'(\alpha)}\right)+3f'\left(\alpha-\frac{2f(\alpha)}{3f'(\alpha)}\right)+5f'\left(\alpha - \frac{f(\alpha)}{f'(\alpha)}\right)\right]} - \alpha\right| \tag{3.11}$$

Since α is the exact solution, $f(\alpha) = 0$. Therefore, from (3.11), $\lim_{n\to\infty}|e_{n+1}| = 0$. That is the error as the number of iterations increases reduces to zero. Hence $|x_{n+1} - \alpha| = 0$, therefore $x_{n+1} - \alpha = 0$ and so $x_{n+1} = \alpha$. Therefore, it is concluded that as the number of iterations increases, the approximate solutions will converge to the exact solution of the given nonlinear equation.

V. Numerical Tests

The proposed method is tested using some benchmark problems in this section. Its performance is compared with the methods of Newton and Eskandari (2017), in terms of the number of iterations and run-time before convergence. The details of the selected benchmark problems and the results of running the algorithms of the iterative methods are presented in Tables 5.1 (A and B). The algorithms were run on MATLAB version 2021a, installed on a personal computer laptop with Processor: intel(R) Core(TM) i5 CPU M 520 @2.40GHz, Installed RAM: 4.00 GB (3.80 GB usable), System type: 64-bit operating system, x64-based processor. The MATLAB codes were programmed to end at the 200th iteration with 15 digits allowed to make room for a reasonable accu-

-racy level.

Table 5.1 (a): Simulation Results 1: Comparison with Newton and Eskandari (2017).

	Problem & Initial Solution	Method	Iterations (n)	Approximate Solution	Absolute Error ($\|f(x_n)\|$)	Run time
1.	$sin^2 x - x^2 + 1 = 0$ $x_0 = 1$	Newton	6	1.404491648215341	9.992×10^{-16}	0.055952
		Eskandari	5			0.040278
		New	6			0.039779
2.	$2x - lnx - 7 = 0$ $x_0 = 1$	Newton	6	4.219906483780382	8.8818×10^{-16}	0.069301
		Eskandari	4			0.102264
		New	4			0.020248
3.	$x^2 - e^x = 0$ $x_0 = 2$	Newton	7	-0.703467422498392	6.6612×10^{-16}	0.039381
		Eskandari	6			0.028641
		New	7			0.063821
4.	$2x^2 - 5x - 2 = 0$ $x_0 = 0$	Newton	5	-0.350781059358212	8.8818×10^{-16}	0.028675
		Eskandari	9			0.041522
		New	7			0.037242
5.	$e^{-x} - \cos x = 0$ $x_0 = 4$	Newton	4	4.721292758847686	1.5092×10^{-16}	0.012462
		Eskandari	3			0.057218
		New	3			0.012024

Table 5.1 (b). Simulation Results 2: Comparison with Newton and Eskandari (2017).

	Problem & Initial Solution	Method	Iteration (n)	Approximate Solution	Absolute Error ($\|f(x_n)\|$)	Run Time (Sec)
6.	$\ln x - e^{-x} = 0$ $x_0 = 2$	Newton	5	1.309799585804150	4.996×10^{-16}	0.037153
		Eskandari	4			0.01456
		New	4			0.018411
7.	$2^x - x^2 = 0$ $x_0 = 0$	Newton	6	- 0.766664695962123	1.1102×10^{-16}	0.043284
		Eskandari	5			0.016359
		New	5			0.017658
8.	$2\sin x - \ln x + 3 = 0$ $x_0 = 6$	Newton	6	5.589235117122464	4.4409×10^{-16}	0.026116
		Eskandari	5			0.004024
		New	4			0.002776
9.	$3^x - 2\cos x = 0$ $x_0 = 4$	Newton	9	0.508033711710655	8.8818×10^{-16}	0.031307
		Eskandari	6	-14.137166851240091	3.3780×10^{-16}	0.024750
		New	8	0.508033711710655	8.8818×10^{-16}	0.007325

Problem & Initial Solution		Method	Iteration (n)	Approximate Solution	Absolute Error ($\|f(x_n)\|$)	Run Time (Sec)
10.	$x^2 - e^x - 3x + 2 = 0$ $x_0 = 0$	Newton	4	0.257530285439861	8.8818×10^{-16}	0.010024
		Eskandari	5			0.004282
		New	5			0.004155

VI. Discussions

The results presented in Tables 5.1A and 5.1B provide some insights into the performance of the new algorithm against a well-known one, such as the Newton's, and against a more recent one, such as the Eskandari (2017), in connection with the ten selected benchmark problems. First of all, concerning approximating the exact solution, all the methods produced the same values, except in an isolated case in Table 5.1B, problem 9, where the method of Eskandari converged to a different root (solution) with a different error margin from the ones achieved by the new method and that of Newton. This may be indication that the method of Eskandari could be sensitive to initial solutions and may converge to a different root than the expected, based on the initial solution chosen. This needs further investigation. The solutions generally provide evidence of the comparability of the effectiveness of the new method to approximate solutions to the same level of accuracy as the two methods named; as well as to converge to the required solution with minimal tolerable error.

In terms of the two main measures of Number of Iterations and Run-Time, we observe from Tables 5.1A and 5.1B the following:

1. Number of iterations in solving Problem 1: The new method (NM) was as good as Newton's (N) with 6 iterations but Eskandari (E) was better with 5 iterations. In run-time, the NM was best with 0.3977
2. Number of iterations in solving Problem 2: The NM was as good as E with 4 iterations and better than N which had 6 iterations. In run-time, the NM was best with 0.0202 seconds.
3. Number of iterations in solving Problem 3: The NM was as good as N with 7 iterations, but E was better with 6 iterations. In run-time, both N and E were better than the NM.
4. Number of iterations in solving Problem 4: The NM was better than E with 7 iterations but N was best with 5 iterations. In run-time, the NM was better than E with 0.037 seconds but N was better than the NM with 0.028 seconds.
5. Number of iterations in solving Problem 5: The NM was as good as E with 3 iterations and better than N which had 4 iterations. In run-time, the NM was best with 0.0120 seconds.
6. Number of iterations in solving Problem 6: The new method (NM) was a s good as Eskandari (E) with 4 iterations and better than Newton (N) which had 5 iterations. In run-time, the NM was as good as E (with time of 0.0145 seconds) and better than N.
7. Number of iterations in solving Problem 7: The NM was as good as E with 5 iterations and better than N with 6 iterations. In run-time, the NM was better than E with 0.037 seconds while N was better than the NM with 0.028 seconds.
8. Number of iterations in solving Problem 8: The NM was the best with 4 iterations followed by E with 5 ite-

-rations and N with 6 iterations. In run-time the NM was best with 0.0027 seconds followed by E with 0.0040 seconds and N with 0.0261 seconds.

9. Number of iterations in solving Problem 9: The NM with 8 iterations was better than N with 9 iterations, but E better than both with 6 iterations. In run-time, the NM was better with 0.0073 seconds than N with 0.0313 and better than E with 0.0217.

10. Number of iterations in solving Problem 10: The NM was as good as E both with 5 iterations with N better with 4 iterations. In run-time, the Nm was best at 0.0041 seconds followed by E with 0.0042 seconds and N with 0.0100 seconds.

The following observations can therefore be gleaned from the results from the two tables summarized in 1 to 10 above, in terms of both number of iterations and run-time:

1. In terms of number of iterations, in 1out of the 10 cases, the NM performed better than both N and E. In terms of run-time, the NM was better than both N and E in 6 out of the 10 cases.

2. In terms of number of iterations, in 5 out of the 10 cases, the NM performed better than N only. In 1out of the 10 cases, NM was better than E only. In terms of run-time, the NM was better than N only in 2 out of the 10 cases and better than E only in 1 out of the 10 cases.

3. In terms of number of iterations, in 2 out of the 10 cases, the NM was as good (i.e., had the same number of iterations) as N only and in 5 out of 10 the NM was as good as E only. In terms of run-time, there were no instances where either N or E was as good as the NM.

4. In terms of number of iterations, in 2 cases out of 10, N only was better than the NM and in 4 cases out of 10, E only was better than the NM. In run-time, however, in 1 case out of 10, N only was better than the NM and in 2 cases out of 10, E only was better than the NM.

Charts to give visual impressions of the above observations (i.e., 1 to 4) are presented in Figure 6.1a, which depicts the observations about the number of iterations, and Figure 6.1b, which depicts the run-time observations. The criteria: BTNE, BTN, BTE, GAN, GAE, GANE, NEB, NB, and EB, respectively refer to: Better Than N and E; Better Than N; Better Than E; Good As N; Good As E; Good As N and E; N and E are Better; N is Better; and E is Better, used to denote comparisons between the NM on one hand and N and/or E on the other.

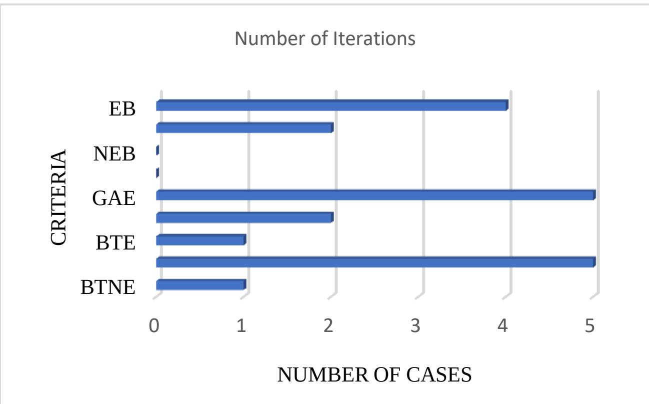

Fig. 6.1A. Comparison of NM with N or E in terms of number of iterations.

From BTNE to GAE, of Figure 6.1a, it is seen that in a large number of cases the NM was either as good as N or as E or better. This reveals the dominance of the NM in terms of using fewer number of iterations to converge to the solutions of the selected problems. The criteria NB and EB indicate the relatively few occasions when N or E were better than the NM. There were no occasions when both N and E were better than the NM at the same time, and there were no occasions when N and E were both as good as the NM; these are denoted by NEB and GANE respectively.

Figure 6.1b shows the dominance of the NM also in yielding the required solution in a relatively shorter time than N and E for the ten benchmark problems. This is revealed by the criteria BTNE, BTN and BTE which indicate that in relatively more of the cases, the NM was either better than N or E, or better than both. In relatively fewer cases either both N and E were better in run-time than the NM or either was better; this is depicted by the criteria NEB, NB, and EB. There were no cases where both N and E had the same run-time as the NM nor where either of them had the same run-time as the NM; these are depicted by GANE, GAN, and GAE.

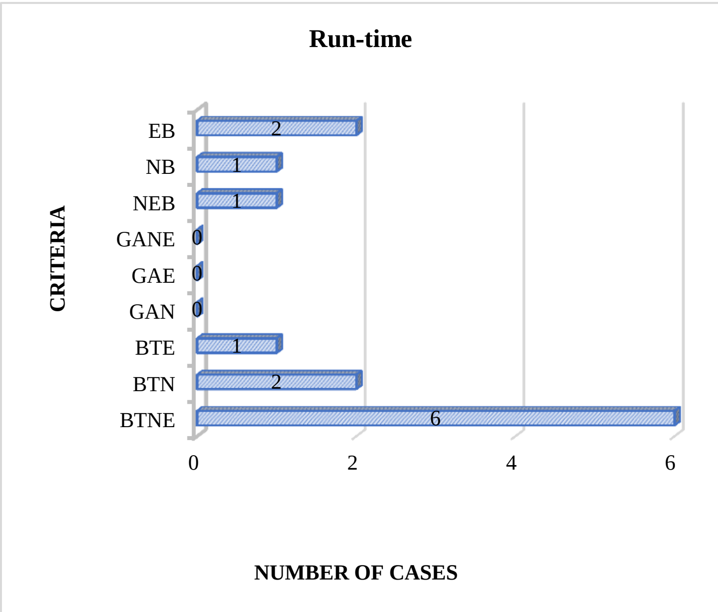

Fig. 6.1B. Comparison of Run-time observations of NM with N and E.

VII. Conclusions

In the real-world context, solution of nonlinear equations in a single variable is non-trivial and require efficient algorithms for their solution. This invariably can only be achieved iteratively which yields an approximate solution. The quality of the approximate solution is therefore of great practical consequence and effort must not be spared to find very good approximate solutions. In this regard, this research has made some contribution in the development of a new iterative scheme using a weighted sum of the Trapezoidal and Simpson's Quadrature rules of integration in conjunction with the Newton's method. The new iterative method has been shown, using benchmark problems, to largely match the Newton's or even exceed it in efficiency. It has also been shown to do as well as or even outperform a leading recent algorithm by Eskandari. The findings from this work, therefore, gives confidence in the performance of the new method in solving nonlinear equations.

Future investigation of the new method could be to assess it in the context of real-world problems; to investigate the reasons for the relative performances of the method against the Newton and Eskandari methods in the context of the problem characteristics. Finally, the idea behind the method may be extended to the development of a solution method for nonlinear systems of equations.

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